The wedge of arbitrage free prices: anything goes*

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ABSTRACT: We show that if K is a closed cone in a finite dimensional vector space X, then there exists a one-to-one linear operator $T\colon X \to C[0,1]$ such that K is the pull-back cone of the positive cone of $C[0,1]$, i.e., $K = T^{-1}(C_+[0,1])$. This problem originated from questions regarding arbitrage free prices in economics.

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1 Introduction

This work deals with cones and wedges of vector spaces. For terminology and notation regarding ordered vector spaces and not explained below we refer the reader to [11], [12] and [8]. For topological vector spaces, we refer to [1] and [10].

A nonempty subset W of a vector space is said to be a **wedge** if it satisfies the following two properties:

1. $W + W \subseteq W$,
2. $\alpha W \subseteq W$ for all $\alpha \geq 0$.

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If, in addition, $W \cap (-W) = \{0\}$, then W is called a **cone**.

Clearly, wedges and cones are convex sets. They are associated with respectively vector pre-orderings and vector orderings of vector spaces. An **ordered vector space** is a vector space X equipped with a cone X_+. The cone X_+ induces a vector ordering $\geq$ on X by letting $x \geq y$ whenever $x - y \in X_+$. An operator $T\colon X \to Y$ between ordered vector spaces is said to be **positive** if $T(X_+) \subseteq Y_+$, i.e., if $x \geq 0$ implies $Tx \geq 0$.

Let $T\colon X \to Y$ be an operator between two vector spaces and let W be a wedge of Y. It is easy to see that the inverse image of W under T is a wedge of X. That is, the set

$$T^{-1}(W) = \big\{x \in X\colon\ T(x) \in W\big\}$$

is a wedge of X. If T is also one-to-one and W is a cone, then the wedge $T^{-1}(W)$ is also a cone of X.

We start with a simple lemma.

Lemma 1. *Let $T\colon X \to Y$ be a one-to-one operator between two vector spaces. If K is a cone of Y, then $T^{-1}(K)$ is a cone of X and the operator $T\colon \big(X, T^{-1}(K)\big) \to (Y, K)$ is a positive operator.*

A cone K of a vector space X is called the **pull-back cone** of the cone of an ordered vector space L if there exists a one-to-one operator $T\colon X \to L$ such that $K = T^{-1}(L_+)$. Alternatively, K is the pull-back of the cone of an ordered vector space L if and only if the ordered vector space (X, K) is order-embeddable in L.

Likewise, a cone K of a topological vector space X is called the **continuous pull-back cone** of the cone of an ordered topological vector space L if there exists a continuous one-to-one operator $T\colon X \to L$ such that $K = T^{-1}(L_+)$. Alternatively, K is the continuous pull-back cone of the cone of a topological ordered vector space L if and only if the topological ordered vector space (X, K) is topologically order-embeddable in L.

As mentioned in the abstract, the objective of this paper is to establish the following basic result. (As usual, $\mathbf{1}$ will denote the constant function one on $[0,1]$, i.e., $\mathbf{1}(t) = 1$ for all $t \in [0,1]$.)

Theorem 2. *Every closed cone of a finite dimensional vector space is the pull-back cone of the* (*standard*) *cone of $C[0,1]$.*

Moreover, if K is a closed and generating cone of a finite dimensional vector space X, then K can be taken to be the pull-back cone of a one-to-one operator $T\colon X \to C[0,1]$ such that $Tu = \mathbf{1}$ for some vector $u \in \mathrm{Int}(K)$.

An interesting consequence follows.

Corollary 3. *A nonempty subset C of $\mathbb{R}^n$ is convex and compact if and only if there exist an $(n+1)$-dimensional subspace E of $C[0,1]$ and a strictly positive linear functional f on E such that C and $E_+ \cap \{x \in E\colon f(x) = 1\}$ are affinely homeomorphic.*[1]

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[1]Two nonempty convex sets A and B (in respectively two topological vector spaces) are affinely homeomorphic if the exists a surjective affine homeomorphism $T\colon A \to B$.

2 Background

2.1 Normal cones

Recall that a subset A of an ordered vector space E is said to be **full** if for each pair $x, y \in A$ the order interval $[x, y] := \{z \in E\colon x \le z \le y\}$ is contained in A.

Definition 4. *A cone K of a topological vector space (E, τ) is said to be **normal** whenever the topology τ has a base at zero consiting of K-full sets (that is, of sets full for the order on E defined by K).*

The notion of normal cone is one of the most useful connections between topology and order of a vector space which implies several nice properties for topological vector spaces ordered by normal cones. In particular, order intervals are topologically bounded, and the existence of a normal cone implies that the topology τ given on the vector space is Hausdorff. Also, if τ is locally convex, then the dual wedge

$$K' =: \{f \in L' \colon\ f(x) \ge 0 \text{ for all } x \ge 0\}$$

is generating in L'. A useful characterization of normal cones is the following:

Theorem 5. *For a cone K of a topological vector space (E, τ) the following statements are equivalent:*

1. *The cone K is normal.*
2. *If two nets $\{y_\alpha\}$ and $\{x_\alpha\}$ of E (with the same index set) satisfy $0 \le_K y_\alpha \le_K x_\alpha$ for each α and $x_\alpha \xrightarrow{\tau} 0$, then $y_\alpha \xrightarrow{\tau} 0$.*

Using this characterization, it is easy to prove the second of the following three basic properties of closed cones in finite dimensional vector spaces:

Lemma 6. *If K is a closed cone of a finite dimensional space E, then:*

1. *The K-order intervals of E are compact.*
2. *The cone K is normal, and*
3. *The dual wedge K' is a cone if and only if K is generating.*

Proof. We shall denote by $\le$ the vector ordering induced by the cone K, that is

$$x \le y \text{ if and only if } y - x \in K\,.$$

(1) The proof is standard. Let $[0, u]$ be a K-order interval. From $[0, u] = K \cap (u - K)$, we see that $[0, u]$ is closed. To see that $[0, u]$ is also norm bounded, assume that a sequence

$\{y_n\} \subset [0,u]$ satisfies $\|y_n\| \to \infty$. Passing to a subsequence, we can assume $\frac{y_n}{\|y_n\|} \to y$. Clearly, $\|y\| = 1$ and so $y \neq 0$. Now from $0 \le y_n \le u$, it follows $0 \le \frac{y_n}{\|y_n\|} \le \frac{u}{\|y_n\|}$, and from the closedness of K and $\frac{u}{\|y_n\|} \to 0$, we see that $0 \le y \le 0$ or $y = 0$, which is impossible. Hence $[0,u]$ is also bounded and thus a compact set.

(2) Assume first that K is generating (that is, $E = K - K$), thus has a nonempty interior. Assume that two sequences $\{y_n\}$ and $\{x_n\}$ of E satisfy $0 \le y_n \le x_n$ for each $n \in \mathbb{N}$ and $x_n \to 0$. Let u be an interior point of K. As $0 \in \operatorname{int}(u - K)$, for large enough n we have $0 \le y_n \le x_n \le u$, hence $0 \le (x_n - y_n) \le x_n \le u$. In view of the compactness of $[0,u]$, passing to a subsequence we can assume that $(x_n - y_n) \to z \in [0,u]$, and so, using again the closedness of K, we see that $-y_n \to 0$.

If the cone K is non-generating, it is at least a generating cone of the vector subspace $K - K$ of E. Since two sequences $\{y_n\}$ and $\{x_n\}$ of E satisfying $0 \le y_n \le x_n$ for each $n \in \mathbb{N}$ and and $x_n \to 0$ are actually sequences lying in the finite dimensional space $K - K$, the desired conclusion follows from the first part of the proof.

(3) Let $x' \in K' \cap (-K')$. If $E = K - K$ then $x' \cdot x = 0$ for all $x \in E$, which proves that $x' = 0$. Conversely, assume that some $x \in E \setminus (K - K)$. As a vector subspace of a finite dimensional vector space, the set $K - K$ is closed. From the separation theorem between a closed convex set and the compact set x, we have $x' \cdot (K - K) = 0$ and $x' \cdot x > 0$ for some $x' \in E'$. If $K' \cap (-K') = \{0\}$, from $x' \cdot (K - K) = 0$ we deduce $x' = 0$, which contradicts $x' \cdot x > 0$. ■

2.2 The Cantor set $\mathcal{C}$ and the space $C(\mathcal{C})$

The Cantor set can be defined as the countable product $\mathcal{C} = \{0,1\}^{\mathbb{N}}$, where the two-point set has the discrete topology. As such, when equipped with the product topology, it is easily seen to be a compact metric space. It can also be thought of as a subset of the real interval $[0,1]$, in an inductive construction where at each step one removes from each closed interval the open middle third-interval. Viewed as a subset of the unit interval, the Cantor set $\mathcal{C}$ is a nowhere dense set of Lebesgue measure zero.

For details and more about the above assertions we refer the reader to [1, pp. 98–101] and to [2, pp. 41–42]. For the rest of our discussion, we need the following well known theorem which can also be found in [1, p. 100].

Theorem 7. *Every compact metric space is the image of the Cantor set under some continuous function.*

Let us recall some terminology. A mapping $f\colon X \to Y$ between two topological spaces is a **topological embedding** if $f\colon X \to f(X)$ is a homeomorphism. Likewise, a linear operator $T\colon X \to Y$ between two ordered vector spaces is a an **order-embedding** if T is one-to-one and if $x \ge 0$ holds in X if and only if $Tx \ge 0$ holds in Y. As usual, $\mathbf{1}_\Omega$ denotes the constant function one on Ω, i.e. $\mathbf{1}(t) = 1$ for all $t \in \Omega$. If $\Omega = [0,1]$, we will simply write $\mathbf{1}$ for $\mathbf{1}_{[0,1]}$.

We shall use below the following easy observation.

Lemma 8. *If $\phi\colon \Omega_1 \to \Omega_2$ is a continuous surjective function between two compact topological spaces, then the mapping $x \mapsto x\circ\phi$ is a norm-preserving order-embedding of $C(\Omega_2)$ into $C(\Omega_1)$ satisfying $\mathbf{1}_{\Omega_2} \circ \phi = \mathbf{1}_{\Omega_1}$.*

Moreover, $x \mapsto x \circ \phi$ is a lattice isomorphism.

Proof. The proof of the first part is straightforward. For the second part note that for each pair $x, y \in C(\Omega_2)$ and each $\omega \in \Omega_1$ we have

$$\begin{aligned}\big[(x\circ\phi)\vee(y\circ\phi)\big](\omega) &= \max\{(x\circ\phi)(\omega),(y\circ\phi)(\omega)\}\\ &= \max\{x(\phi(\omega)),y(\phi(\omega))\} = (x\vee y)(\phi(\omega))\\ &= \big[(x\vee y)\circ\phi\big](\omega)\,.\end{aligned}$$

Thus, $(x\circ\phi)\vee(y\circ\phi) = (x\vee y)\circ\phi$ and hence $x \mapsto x\circ\phi$ is a lattice isomorphism. ■

The next lemma is an immediate consequence of Theorem 7 and Lemma 8.

Lemma 9. *If Ω is a compact metrizable topological space, then there exists a norm-preserving order-embedding of $C(\Omega)$ into $C(\mathcal{C})$ that carries $\mathbf{1}_\Omega$ to $\mathbf{1}_\mathcal{C}$.*

Our major intermediate result is the following:

Lemma 10. *There is a norm-preserving order-embedding of $C(\mathcal{C})$ into $C[0,1]$ that maps $\mathbf{1}_\mathcal{C}$ to $\mathbf{1}$.*

In particular, if Ω is any compact metrizable topological space, then there exists a norm-preserving order-embedding of $C(\Omega)$ into $C[0,1]$ in such a way that $\mathbf{1}_\Omega$ is mapped to $\mathbf{1}$.

Proof. Recall that the complement of the Cantor set $\mathcal{C}$ can be written as a countable union of pairwise disjoint open intervals. That is, we can write $[0,1]\setminus\mathcal{C} = \bigcup_{n=1}^{\infty}(a_n,b_n)$, where $(a_n,b_n)\cap(a_m,b_m) = \emptyset$ for $n \neq m$. Now each $x \in C(\mathcal{C})$ can be extended to a function $\widehat{x} \in C[0,1]$ by extending the graph of x on each open interval (a_n,b_n) to coincide with the graph of the line segment joining the points $(a_n, x(a_n))$ and $(b_n, x(b_n))$. That is, for each $a_n < t < b_n$ we let

$$\widehat{x}(t) = \tfrac{x(b_n)-x(a_n)}{b_n-a_n}(t-a_n) + x(a_n)\,.$$

Some easy verifications show that:

(a) $\widehat{x}$ is a continuous function.

(b) If $x = \mathbf{c}$, the constant function c, then $\widehat{x} = \mathbf{c}$. In particular, $\widehat{\mathbf{1}}_\mathcal{C} = \mathbf{1}$.

(c) $\widehat{x} \geq 0$ holds in $C[0,1]$ if and only if $x \geq 0$ holds in $C(\mathcal{C})$.

(d) If $x, y \in C(\mathcal{C})$ and $\lambda \in \mathbb{R}$, then $\widehat{x+y} = \widehat{x} + \widehat{y}$ and $\widehat{\lambda x} = \lambda\widehat{x}$.

(e) $\max_{t\in\mathcal{C}} |x(t)| = \max_{t\in[0,1]} |\widehat{x}(t)|$.

The above properties show that $x \mapsto \widehat{x}$ is a norm-preserving order-embedding of $C(\mathcal{C})$ into $C[0,1]$ satisfying $\widehat{\mathbf{1}}_{\mathcal{C}} = \mathbf{1}$.

The last part follows easily from the above conclusion and Lemmas 8 and 9. ■

3 The Proof of Theorem 2

We shall actually prove a more general result from which Theorem 2 is a simple consequence.

Theorem 11. *For a separable ordered Banach space E with a closed normal positive cone K we have:*

(a) *There is a one-to-one, order-preserving, linear operator $T: E \to C[0,1]$.*

(b) *If, in addition, K satisfies $\overline{K-K} = E$, then the operator T [from E onto $T(E)$] is also a homeomorphism.*

Proof. (a) Let $\Omega := \{x' \in K' \colon \ \|x'\| \le 1\}$. From the separability of E and the Alaoglu–Bourbaki Theorem, it follows that Ω equipped with its w^*-topology is a compact metrizable topological space (see [1, Theorem 6.30, p. 239]).

Now define the mapping $R\colon E \to C(\Omega)$ by letting $(Rx)(\omega) = \omega(x)$ for all $x \in L$ and all $\omega \in \Omega$. It should be clear that R is a linear operator. The normality of the cone K implies that the wedge K' is generating in L'. This guarantees that a linear functional on E' is the zero functional if and only if it vanishes on Ω. Consequently, from

$$Rx = 0 \iff \omega(x) = 0 \ \text{for all} \ \omega \in \Omega \iff x = 0\,,$$

it follows that R is one-to-one. Moreover, using that K is closed, we see that

$$\begin{aligned} Rx \ge 0 \ &\iff \ \omega(x) \ge 0 \ \text{for all} \ \omega \in \Omega \\ &\iff \ x'(x) \ge 0 \ \text{for all} \ x' \in K' \\ &\iff \ x \in K'' = K\,, \end{aligned}$$

where K'' is the dual cone in E of K' with respect to the dual system $\langle E, E' \rangle$ (that $K'' = K$ follows from the bipolar theorem). This implies that $R\colon E \to C(\Omega)$ is an order-embedding. Now apply Lemma 10.

(b) Notice first that for each $x \in E$ we have $\|Rx\|_\infty = \sup_{\omega\in\Omega} |\omega(x)| \le \|x\|$. Now assume $\overline{K-K} = E$. As in the finite dimensional case, we can easily see that K' is a closed cone, generating since K is normal and E locally convex. It then follows from a theorem of Andô [9] (see also [8]) that $\Omega - \Omega$ is a 0-neighborhood for the norm topology of E'. This implies that there exists some $\rho > 0$ such that for each x' such that $\|x'\| \le 1$ there exist $y', z' \in \Omega$ satisfying $\|y'\| \le \rho$, $\|z'\| \le \rho$, and $x' = y' - z'$. In particular, for each x' in the unit ball U' of E' and each $x \in E$ we have

$$|x'(x)| \le \rho\left|\tfrac{y'}{\rho}(x)\right| + \rho\left|\tfrac{z'}{\rho}(x)\right| \le 2\rho\|Rx\|_\infty .$$

We have also $\|x\| = \sup_{x' \in U'} |x'(x)| \le 2\rho\|Rx\|_\infty$. Therefore, for each $x \in E$ we have

$$\frac{1}{2\rho}\|x\| \le \|Rx\|_\infty \le \|x\|$$

so that (in this case) R is also a topological order-embedding. To complete the proof now note that (according to Lemma 10) $C(\Omega)$ is topologically order-embeddable in $C[0,1]$. ■

To complete the section, we show how Theorem 2 can be deduced from the previous one.

Corollary 12. *Every closed cone K of a finite dimensional vector space E is order-embeddable in $C[0,1]$. If, moreover, K is generating (that is, if $E = K - K$), then T, the linear operator which topologically order-embeds E into $C[0,1]$, can be chosen so as $T(u) = \mathbf{1}$ for some $u \in \operatorname{int} K$.*

Proof. A finite dimensional (real) vector space is obviously a separable Banach space. Assume now that $E = K - K = \overline{K - K}$. The function $\mathbf{1}$ is an order-unit thus an interior point of $C_+[0,1]$. Thus $T^{-1}(\mathbf{1})$ is an interior point of K. ■

4 The wedge of arbitrage free prices

The present work originated from questions in financial economics. It is motivated by the counter example in [7] and the resolution of the economic problem highlighted by the example in [4, 5, 6]. We briefly illustrate this connection below.

We consider the standard two-period securities model. That is, we suppose that there are two periods 0 and 1 ("today" and "tomorrow"). In period 0 everything is known while in period 1 there is uncertainty. The uncertainty is described by a probability space $(\Omega, \mathcal{B}, P)$. We view the vector space $L_0(\Omega, \mathcal{B}, \pi)$ of all equivalence classes of measurable real functions on Ω as the asset space. The elements of $L_0(\Omega, \mathcal{B}, \pi)$ are called **assets**.

We assume that in our market today there is a finite number of non-redundant (i.e., linearly independent) assets $f_1, f_2, \dots, f_n$ that can be purchased by the consumers. A portfolio is a vector $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$. With each portfolio θ we consider the asset $T\theta$, defined for each $s \in \Omega$ by

$$[T\theta](s) = \sum_{i=1}^{n} \theta_i f_i(s) . \qquad (\star)$$

The interpretation of $[T\theta](s)$ is the following: If a consumer holds the portfolio θ and the materialized state of the world tomorrow is s, then the value (payoff) of the portfolio θ is precisely $[T\theta](s)$.

It is not difficult to see that $(\star)$ defines a one-to-one linear operator $T\colon \mathbb{R}^n \to L_0(\Omega, \mathcal{B}, \pi)$. This operator is called the *payoff operator* and its range is precisely the subspace M of $L_0(\Omega, \mathcal{B}, \pi)$ spanned by the available assets $f_1, f_2, \ldots, f_n$.

An asset price is also a vector $q \in \mathbb{R}^n$. It is called **arbitrage free** if for each portfolio $\theta \in \mathbb{R}^n$ satisfying $[T\theta](s) \geq 0$ for almost all $s \in \Omega$ and $P(\{s \in \Omega\colon [T\theta](s) > 0\}) > 0$ we have $q \cdot \theta > 0$. Let $\mathcal{A}$ be the set of arbitrage free prices. Notice that $\mathcal{A}$ is an **open wedge** i.e., it is an open convex set that satisfies $\alpha q \in \mathcal{A}$ for all $\alpha > 0$ and $q \in \mathcal{A}$. In the special case where $\mathcal{A}$ satisfies $\mathcal{A} \cap (-\mathcal{A}) = \emptyset$ we say that $\mathcal{A}$ is an **open cone**. The notion of arbitrage free prices is of enormous importance in financial economics.

The set of arbitrage free prices $\mathcal{A}$ is never empty because the set

$$K = \{\theta \in \mathbb{R}^n\colon\ [T\theta](s) \geq 0 \text{ a.e.}\} = T^{-1}(L_0^+)$$

is always a closed cone. The cone K is called the **portfolio cone** of the assets $f_1, f_2, \ldots, f_n$. It induces a vector ordering on E called **portfolio dominance**; see [3]. The set of arbitrage free prices $\mathcal{A}$ is the interior of the dual

$$K' = \{q \in \mathbb{R}^n\colon\ q \cdot \theta \geq 0 \text{ for all } \theta \in K\}\,.$$

Now we consider the space $C[0,1]$ as canonically embedded in L_0 with the Lebesgue measure. Theorem 2 can easily be re-stated as follows.

Theorem 13. *If $\mathcal{A}$ is a non-empty open wedge in $E = \mathbb{R}^n$, then there exist non-redundant assets $f_1, f_2, \ldots, f_n$ in $C[0,1]$ such that the set of arbitrage free prices is $\mathcal{A}$.*

If $\mathcal{A}$ is an open cone, then f_1 can be chosen to be the constant function (*bond*) $\mathbf{1}$ *satisfying $f_1(s) = 1$ for all $s \in [0,1]$.*

Proof. Since $\mathcal{A}$ is an open wedge, its dual is a closed cone K to which we can apply Theorem 2. Let $T\colon \mathbb{R}^n \to C[0,1]$ be a one-to-one operator such that $K = T^{-1}(C_+[0,1])$. Take for assets $f_1, f_2, \ldots, f_n$ any basis of $T(K)$. The set of arbitrage free prices is the interior of K', i.e., the set $\mathcal{A}$. To see the equivalence between the respective conditions in Theorem 13 and Theorem 2 that $\mathcal{A}$ is an open cone and that K is generating, apply Lemma 6. ■

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