EQUILIBRIUM OF INCOMPLETE MARKETS WITH MONEY AND INTERMEDIATE BANKING SYSTEM

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Abstract. This paper studies a simple stochastic two-period general equilibrium exchange model with money, an incomplete market of nominal assets, and a competitive banking system, intermediate between consumers and a Central Bank. There is a finite number of agents, consumers and banks. Default is not permitted. The public policy instruments are, besides real taxes implicit in the model, public debt and creation of money both implemented at the first period. The equilibrium existence is established under a "Gains to trade" hypothesis and the assumption that banks have a non zero endowment of money at each date-event of the model.

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1. Introduction

Various contributions (among others Magill and Quinzii [18], Drèze and Polemarchakis [6], and Dubey and Geanakoplos [8]) have introduced money in a simple two-period stochastic general equilibrium model with financial markets. In these models, cash-in-advance constraints within each period (each date-event at the second period) explain the role of money by short term bank loans which provide within-period liquidity to individual agents, while financial markets implement for these agents transfers across periods. The money permitting transactions is injected at each date-event into the economy by a monetary authority. This money supply together with the corresponding interest rates are, according to the chosen point of view, endogenous or exogenously specified variables in the definition of monetary equilibrium. Namely, either interest rates form endogenously in equilibrium to clear the loan markets so far the Central Bank is committed to specified quantities of money, or the different stocks of money supplied in the loan markets are endogenous, setting the interest rates as exogenous. A common objective of both frameworks is to analyze fiscal and monetary policy.

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In Drèze and Polemarchakis [6], financial markets are complete markets of purely financial assets, more precisely complete markets of elementary securities, each one with payoff of one unit of revenue at a date-event of the second period. Market completeness allows for the definition of a consolidated economy in terms of present-value prices and monetary balances which simplifies the monetary equilibrium existence problem, but the model can be extended in principle to incomplete asset markets. In Magill and Quinzii [18], financial markets of nominal assets are incomplete. In their paper, the role of the public authority is not only to inject into the economy the supply of money permitting transactions but, under the name of Central Exchange, to perform the function of marketing the agents' endowments under the simplifying assumption that selling and buying commodity prices faced by individual agents are proportional. Under this assumption, the existence of a monetary equilibrium is proved using methods which extend to this framework methods used in the GEI equilibrium existence problem. Finally in Dubey and Geanakoplos [8], asset markets are incomplete and the equilibrium existence problem is dealt with, without intervention of Central Exchange. Besides a greater generality in the definition of assets, which do not need be nominal assets, and of the bonds sold in counterpart of bank loans, the main difference with the previous models, where outside money[1] was explicitly ruled out, is in the hypothesis that individual agents have at each date-event private endowments of money. The presence of this outside money plays a role in the monetary equilibrium existence proof. Equilibrium exists if the total private endowment of money is strictly positive at the first period and if the government money supply is also strictly positive on all monetary markets. Without private endowment of money, equilibrium existence may fail, especially if assets are not nominal.

In the present paper, as Tsomocos [19], we add to the standard general equilibrium exchange model with incomplete markets and money a competitive banking system intermediate between agents and the Central Bank. Our modeling choices, which will be made precise in Section 2, parallel the ones of Tsomocos [19] and Goodhart, Sunirand and Tsomocos [12], with several differences that the precise description of the model will make clear. Consumers and commercial banks have at each date-event a contingent endowment of (outside) money that one can interpret, at least for consumers, as a government transfer or, preferably, as the private inheritance from the (unmodelled) past. Consumers and banks have access to the financial market. At each date-event, they are submitted to a cash-in-advance constraint that they satisfy using the different monetary markets. More precisely, at each date-event, banks extend credit to consumers via the short term credit market. In counterpart, banks may borrow from consumers in the short term and from the Central Bank in the long term via the interbank credit market.

As a first approach to the equilibrium existence problem, default is not permitted. When studying the role of the banking system in an economy without production and whose only available goods are non-durable, absence of default has probably some realism. Also, while Tsomocos considers assets with payoffs denominated in commodity bundles as well as in money, following Magill and Quinzii [18], Drèze and Polemarchakis [6], we restrict ourselves to financial markets of purely nominal assets. And we model here the public debt as an explicit government intervention (namely as an initial positive supply of bonds) on the financial market. Together with the amount of created money and the specification of taxes, implicitly described in our model by the definition of (after taxes) real endowments of agents, the amount of the public debt is one of the instruments of the fiscal-financial-monetary policy of government. Besides its budget constraint (and in particular the

[1] *Inside* money is money created in the banking activity. *Outside* money does not bear any other commitment than the persistence of its existence.

obligation of paying back the public debt at the second period), the government has no optimizing behavior, contrary to the consumers who maximize intertemporal utility under budget constraints and to the commercial banks which maximize each a utility function of their positive balance at the end of each date-event. As in Tsomocos [19], the assumption that that consumers bid at the first period for shares of commercial banks which determine the share of the final bank balances they receive at the different states of the last period closes the model. Equilibrium is defined as a collection of prices and actions such that consumers and banks optimize their utility under their budget constraint and all markets clear.

We focus here on the equilibrium existence problem when the level of each public policy instrument is exogenously specified. The equilibrium existence theorem we prove requires, besides usual assumptions of consumers'survival and continuity, strict monotonicity and strict quasi-concavity of utility functions, the presence in the model of both inside and outside money and the assumption that, roughly speaking, autarky is never optimal at any date-event for consumers. Some form of such an hypothesis, usually known as "gain-to-trade hypothesis", is used in any monetary general equilibrium model in order to guarantee that money is effectively used at equilibrium.

Our equilibrium existence proof is classical in its general philosophy but by no means in its specification. Along with the market clearing relations, monetary regulations of the banking activity (in use in every "real" economy) allow for the definition of a compact economy. Equilibrium in the compact economy arises from a simultaneous optimization process performed by consumers, banks and two kind of auctioneers. The first one, as the Walrasian auctioneer of the Arrow–Debreu model [1], sets here not only commodity and asset prices but also the different interest rates in reaction to the market actions of the different agents. According to an idea of Gourdel–Triki [14], the second auctioneer sets state prices for money at each date-event so as to allow the supply of money to meet at equilibrium the cash in advance needs of consumers and banks. Finally, the equilibrium of the compact economy is proved to be an equilibrium of the original economy.

The paper is organized as follows. Section 2 sets forth the model and its different assumptions. Section 3 is devoted to the equilibrium existence proof. We emphasize that the equilibrium existence could not be deduced from any existing result. Introducing the possibility of default under different settings as well as studying the relations between existence of equilibrium with and without default will be the object of future research.

2. The model

We consider a closed competitive model with a finite set I of consumers, a finite set B of commercial banks and a Central Bank. There are two time periods $t \in \{0, 1\}$ and a finite set S of states of the world at time $t = 1$. At time $t = 0$, the state of the world, denoted $s = 0$, is known with certainty and we set $S^* = S \cup \{0\}$. As in the classical GEI model, there is at each state $s \in S^*$ a spot market for a finite set L of goods, and, at time 0, a market for a finite set J of nominal assets whose returns are paid at time 1 contingent on the realized state of the world. The j^{th} column of a $(S \times J)$-matrix R defines the return of asset j in each state s of the period $t = 1$ denominated in money. One of the assets (say the first one) is a public bond whose return is 1 in every state of nature at time 1. This bond is in positive supply z_1^G, an exogenous quantity which represents the initial public debt fixed by the authority, to be repaid at time $t = 1$. The assumption that consumers and banks have no endowment in assets and no short sale constraint completes the description of the financial structure of the economy. In addition, there is, at time 0, an auction market open to all consumers for the profit shares of the commercial banks.

Money is the stipulated means of exchange. Every expenditure of consumers or banks is submitted to a cash-in-advance constraint which expresses that it should be paid with the money at hand at the time of the transaction. On the other hand, transactions are facilitated by the existence of a system of short term deposits and loans from consumers to banks at each state of the world, and by a system of long term deposits and loans from banks to the interbank credit market. The Central Bank conducts its monetary policy through creation of money. A regulation authority fixes requirements for the commercial banks.

Each consumer i has $\mathbb{R}_+^{L(1+S)}$ as consumption set. He is characterized by a contingent endowment of goods $e^i \in \mathbb{R}_+^{L(1+S)}$, a contingent endowment of money $e^i_m \in \mathbb{R}_+^{1+S}$ and by a utility function $u^i\colon \mathbb{R}_+^{L(1+S)} \to \mathbb{R}$ that represents as well his intertemporal preferences on consumption of goods as his attitude toward uncertainty.

Each commercial bank b is characterized by a contingent initial endowment in money $e^b_m \in \mathbb{R}_+^{1+S}$ and by an objective function $u^b\colon \mathbb{R}^{1+S} \to \mathbb{R}$ of the vector $\big(\pi^b(s)\big)_{s\in S^*}$ of its monetary holdings at the end of the period in each state of the world. The function u^b describes as well the intertemporal preferences of b on monetary holdings as its attitude toward uncertainty. For example, following the von Neumann–Morgenstern definition, one can assume that $u^b\Big(\big(\pi^b(s)\big)_{s\in S^*}\Big) = \pi^b(0) + \lambda^b \sum_{s\in S} \alpha^b_s \pi^b(s)$, where λ^b is a discount factor and $(\alpha^b_s)_{s\in S}$ is a probability distribution on S. Financial requirements are modelled through reserve and capital ratios $(k^1_s)_{s\in S^*}$ and $(k^2_s)_{s\in S^*}$ to be precise later.

As consumers can consume, buy or sell assets, deposit and lend money in the short term credit market, bid for shares in the ownership of commercial banks, and transfer money from time 0 to period 1, the choice set of consumer i is $\Sigma^i = \mathbb{R}_+^{LS^*} \times \mathbb{R}^J \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+^B \times \mathbb{R}_+$. As commercial banks can buy or sell assets, admit short term deposits and extend short term credits to consumers, deposit and lend money in the interbank credit system, choose to hold money at the end and in each state of each period, the choice set of each commercial bank is $\Sigma^b = \mathbb{R}^J \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+^{S^*} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+^{S^*}$.

Let us denote by $z^G = (z^G_1, 0, \ldots, 0)$ the public portfolio and by M^{CB} the money created (or destroyed) by the Central Bank. The previous data define the economy:

$$\mathcal{E} = \Big((\Sigma^i, u^i, e^i, e^i_m)_{i\in I}, (\Sigma^b, u^b, e^b_m)_{b\in B}, R, (k^1_s, k^2_s)_{s\in S^*}, z^G, M^{CB}\Big).$$

The functioning of $\mathcal{E}$ is as follows.

At each state $s \in S^*$, the consumer i consumes $x^i(s) \in \mathbb{R}_+^L$, borrows the quantity $\mu^i(s) = \sum_{b\in B} \mu^i_b(s) \in \mathbb{R}_+$ or deposit the quantity $\delta^i(s) = \sum_{b\in B} \delta^i_b(s) \in \mathbb{R}_+$ in the short term consumer credit market. In addition, at time 0, i chooses a portfolio $z^i \in \mathbb{R}^J$ and spends the quantities $(\upsilon^i_b)_{b\in B} \in \mathbb{R}_+^B$. These quantities determine its shares

$$\theta^i_b = \begin{cases} \frac{\upsilon^i_b}{\sum_{j\in I} \upsilon^j_b} & \text{if } \sum_{j\in I} \upsilon^j_b > 0 \\ 0 & \text{if } \sum_{j\in I} \upsilon^j_b = 0 \end{cases}$$

in the profit of each bank b at period 1.

To summarize, given price systems $p \in \mathbb{R}_+^{L(1+S)}$ for commodities and $q \in \mathbb{R}_+^J$ for assets, given short term interest rates $\big(r_d(s), r(s)\big)_{s\in S^*} \in \mathbb{R}_+^{2S^*}$ for deposits and loans of consumers,[2], given also

[2] In view of the writing of the equilibrium model and the assumptions which will be done later on the utility functions and the return matrix, commodity and asset prices and all short term and long term interest rates are assumed to be non-negative.

the vectors of distributed profits $\big((\pi^b(s))_{b\in B}$ of commercial banks at each state $s \in S$ and the quantities bidden by the other consumers in the equity market for commercial banks, the budget set of consumer i in his choice set Σ^i

$$B^i\Big(p, q, r_d, r, \big(\pi^b(s)_{s\in S}\big)_{b\in B}, (\upsilon^{i'_b})_{i'\neq i,\ b\in B}\Big)$$

is determined by the following constraints at period 0 and in each state s of period 1. As all transactions are made against money, agents cannot use for their purchases the receipts from their sales. Cash-in-advance constraints (1) and (3) below express this hypothesis. For the sake of notational ease, the positive part (resp. the negative part) of any vector y is indifferently denoted y^+ or y_+ (resp. y^- or y_-).

At time $t = 0$,

$$p(0)\cdot\big(x^i(0)-e^i(0)\big)^+ + q\cdot z^i_+ + \delta^i(0) + \sum_{b\in B}\upsilon^i_b \leq\ e^i_m(0) + \frac{\mu^i(0)}{1+r(0)} \tag{1}$$

$$p(0)\cdot\big(x^i(0)-e^i(0)\big) + q\cdot z^i + \frac{r(0)}{1+r(0)}\mu^i(0) + \sum_{b\in B}\upsilon^i_b + \pi^i(0) \leq\ e^i_m(0) + r_d(0)\delta^i(0) \tag{2}$$

For symmetry with banks, we denote $\pi^i(0)$ the positive amount of money that consumer i chooses to transfer from period 0 to period 1. The quantity $\pi^i(0)$ is not an argument *per se* of the utility function of i, but $\pi^i(0) > 0$ may facilitate a greater consumption of i in every state of $t = 1$.

$$p(s)\cdot\big(x^i(s)-e^i(s)\big)^+ + R(s)\cdot z^i_- + \delta^i(s) \leq\ e^i_m(s) + \pi^i(0) + \frac{\mu^i(s)}{1+r(s)} \tag{3}$$

$$p(s)\cdot\big(x^i(s)-e^i(s)\big) + \frac{r(s)}{1+r(s)}\mu^i(s) \leq e^i_m(s) + \pi^i(0) + R(s)\cdot z^i + r_d(s)\delta^i(s) + \sum_{b\in B}\theta^i_b\pi^b(s). \tag{4}$$

Each consumer chooses $\sigma^i = \big(x^i, z^i, \delta^i, \mu^i, (\upsilon^i_b)_{b\in B}, \pi^i(0)\big) \in \Sigma^i$, so as to maximize his utility function $u^i(x^i)$ in this budget set.

Symmetrically, at each state $s \in S^*$, the bank b extends a total amount $m^b(s)$ of short term credits to consumers, admits from them a total amount of short term deposits $d^b(s)$. At time 0, b chooses a portfolio $z^b \in \mathbb{R}^J$, borrows the amount of money μ^b and loans the amount of money δ^b in the interbank credit market, issues the equities $\upsilon^b = \sum_{i\in I}\upsilon^i_b$ and transfers the positive amount of money $\pi^b(0)$ from period 0 to the different states of period 1. Notice that in our model, commercial banks have a passive role in the equity markets which automatically clears.

Given the price system $q \in \mathbb{R}^J_+$ for assets, given short term interest rates $\big(r_d(s), r(s)\big)_{s\in S^*} \in \mathbb{R}^{2S^*}_+$ set respectively for deposits and loans in the consumer credit market, given the long term interest rate $\rho \in \mathbb{R}_+$ in the interbank credit market for deposits and loans, given the quantities υ^i_b bidden by the consumers in the equity market for banks, the budget set of bank b

$$B^b\Big(q, r_d, r, \rho, (\upsilon^i_b)_{i\in I}\Big)$$

is defined in its choice set Σ^b by the following constraints at time 0 and in each state s of period 1.

For $s = 0$,

$$\delta^b \leq e_m^b(0) \tag{5}$$

$$q \cdot z_+^b + \delta^b + m^b(0) + \leq e_m^b(0) + \frac{d^b(0)}{1 + r_d(0)} + \frac{\mu^b}{1+\rho} + \sum_{i \in I} \upsilon_b^i \tag{6}$$

$$q \cdot z^b + \delta^b + \frac{r_d(0)}{1 + r_d(0)} d^b(0) + \pi^b(0) \leq e_m^b(0) + r(0) m^b(0) + \frac{\mu^b}{1+\rho} + \sum_{i \in I} \upsilon_b^i \tag{7}$$

$\pi^b(0)$ denotes the monetary holding of b at the end of $t = 0$ that bank b chooses to transfer from period 0 to the different states of period 1. Recall that this quantity is an argument of the utility function of the bank b.

For each s of period 1,

$$R(s) \cdot z_-^b + m^b(s) \leq e_m^b(s) + \pi^b(0) + \frac{d^b(s)}{1 + r_d(s)} \tag{8}$$

$$\frac{r_d(s)}{1 + r_d(s)} d^b(s) + \mu^b + \pi^b(s) \leq e_m^b(s) + \pi^b(0) + R(s) \cdot z^b + r(s) m^b(s) + (1+\rho)\delta^b \tag{9}$$

In addition at time $t = 0$ and at each state s of time $t = 1$, each commercial bank b has to satisfy the following capital and reserve requirements , where the strictly positive coefficients k_s^1 and k_s^2 are fixed by the government (or the Cental Bank)

$$k_s^1 d^b(s) \leq \delta^b \ \forall s \in S^*; \tag{10}$$

$$k_s^2 m^b(s) \leq e_m^b(s) \ \forall s \in S^*. \tag{11}$$

In view of (5), and depending on the size of k_s^1 (which may be small), the reserve requirements limit the short term deposits admitted by banks. Depending on the size of k_s^2 (which may be small), the capital requirements limit the short term credits that banks can extend.

$\pi^b(s)$ denotes the positive monetary holding of b at the end of period $t = 1$ in each state $s \in S$, to be distributed among consumers following the profit shares θ_b^i.

Each bank chooses $\sigma^b = \left(z^b, d^b, m^b, \delta^b, \mu^b, \pi^b\right) \in \Sigma^b$ so as to maximize its objective function $u^b\left((\pi^b(s)_{s \in S^*})\right)$ in its budget set.

The government has at time $t = 0$ a revenue equal to $q \cdot z^G$ and spends at each state $s \in S$ of period $t = 1$ the amount $R(s) \cdot z^G = z_1^G$. This revenue and this expenditure have to be taken in account when the addition of the end-of period constraints describes the Walras law for this economy.[3] This writing emphasizes the idea that the reimbursement of the public debt is charged on the revenue of agents at time $t = 1$.

Central Bank has to verify his balance sheet according to money supply M^{CB}.

Definition 2.1. A **monetary equilibrium** of $\mathcal{E}$ is a collection $\left((p, q, r_d, r, \rho), (\sigma^i)_{i \in I}, (\sigma^b)_{b \in B}\right) \in \mathbb{R}^{LS^*} \times \mathbb{R}^J \times \mathbb{R}^{S^*} \times \mathbb{R}^{S^*} \times \mathbb{R} \times \prod_{i \in I} \Sigma^i \times \prod_{b \in B} \Sigma^b$ where

[3]It should be understood that the project of Government and real taxes on consumers are implicit in the definition of z^G and of (after taxes) contingent endowments of consumers.

(1) For each $i \in I$, $\sigma^i \in \operatorname{argmax}\{u^i(\widetilde{x}^i)\colon \widetilde{\sigma}^i = (\widetilde{x}^i, \widetilde{z}^i, \widetilde{\delta}^i, \widetilde{\mu}^i, (\widetilde{v}^i_b)_{b\in B}) \in B^i(p, q, r, r_d, (\sigma^b)_{b\in B})\}$,
(2) For each $b \in B$, $\sigma^b \in \operatorname{argmax}\{u^b(\widetilde{\pi}^b(s)_{s\in S^*})\colon \widetilde{\sigma}^b = (\widetilde{z}^b, \widetilde{d}^b, \widetilde{m}^b, \widetilde{\delta}^b, \widetilde{\mu}^b) \in B^b(q, r_d, r, \rho, (\sigma^i)_{i\in I})\}$,
(3) All markets clear:
  (a) $\sum_{i\in I}(x^i(s) - e^i(s)) = 0$, $\forall s \in S^*$ i.e. all commodity markets clear,
  (b) $\sum_{i\in I} z^i + \sum_{b\in B} z^b = z^G = (z^G_1, 0, \dots, 0)$, i.e. asset market clears,
  (c) $\big(1 + r(s)\big) \sum_{b\in B} m^b(s) = \sum_{i\in I} \mu^i(s)$ $\forall s \in S^*$, i.e. credit market clears,
  (d) $\sum_{b\in B} d^b(s) = \big(1 + r_d(s)\big) \sum_{i\in I} \delta^i(s)$ $\forall s \in S^*$, i.e. deposit market clears,
  (e) $\sum_{b\in B} \mu^b = (1+\rho)(\sum_{b\in B} \delta^b + M^{CB})$, i.e. the balance sheet of Central Bank is verified,
  (f) For each bank b, $\sum_{i\in I} \theta^i_b = 1$, i.e. the equity markets for ownership shares of banks clear.

Definition 2.2. We will call **feasible allocation** any collection $\big((\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big) \in \prod_{i\in I} \Sigma^i \times \prod_{b\in B} \Sigma^b$ satisfying the market clearing relations (3) of the previous definition.

Let us set $e = \sum_{i\in I} e^i$, $M(0) = \sum_{i\in I} e^i_m(0) + \sum_{b\in B} e^b_m(0)$, $\forall s \in S$, $M(s) = \sum_{i\in I} e^i_m(s) + \sum_{b\in B} e^b_m(s)$. We will prove the existence of a monetary equilibrium under the following assumptions on $\mathcal{E}$:

A.1: For each $i \in I$,
  (1) The consumption set is $\mathbb{R}^{L(1+S^*)}_+$, $e^i \in \mathbb{R}^{L(1+S^*)}_+$,
  (2) The utility function $u^i\colon \mathbb{R}^{L(1+S)}_+ \to \mathbb{R}$ is continuous, strictly monotone and strictly quasi-concave[4],
  (3) $e^i_m = \big(e^i_m(s)\big)_{s\in S^*} \geq 0$.

A.2: For each $b \in B$,
  (1) $e^b_m = \big(e^b_m(s)\big)_{s\in S^*} \geq 0$,
  (2) The objective function $u^b\colon \mathbb{R}^{1+S} \to \mathbb{R}$ is continuous, strictly monotone and strictly quasi-concave.

A.3: On the financial side,
  (1) The return matrix R has full rank and we assume $R \geq 0$, $R^1 = (1, 1, \dots, 1)$ (the first asset has 1 for return in every state $s \in S$),
  (2) Each agent (consumer or bank) has $\mathbb{R}^J$ as portfolio set,
  (3) The total supply of the first asset, equal to z^G_1, is positive: $z^G_1 > 0$; the other assets are in zero supply.

A.4: Survival and presence of inside and outside money in the economy:
  (1) $\sum_{i\in I} e^i(0) \gg 0$ and for each $i \in I$, for every $s \in S$, $e^i(s) \gg 0$,
  (2) For each $b \in B$, for every $s \in S^*$, $e^b_m(s) > 0$,
  (3) $M^{CB} > 0$,
  (4) $M(s) = M(s')$ $\forall s, s' \in S$.

A.5: Gains to trade:
Every consumption allocation $(x^i)_{i\in I} \in \mathbb{R}^{LS^*I}$ that satisfies, for some $s \in S^*$, $x^i(s) = e^i(s)$ $\forall i \in I$ permits, in this state, at least δ-gain to trade where

$$\delta = \frac{M(0) + M(s)}{M^{CB}},$$

[4]By strict quasi-concavity of a function u^i we mean that u^i is quasi-concave and that

$$u^i(\widetilde{x}^i) > u^i(x^i) \Longrightarrow u^i\big(\lambda x^i + (1-\lambda)\widetilde{x}^i\big) > u^i(x^i)\ \forall \lambda\colon 0 < \lambda < 1.$$

that is, there exist trades $\left(\tau^i(s)\right)_{i\in I} \in \mathbb{R}^L$, such that $\sum_{i\in I}\tau^i(s) = 0$, and for each $i \in I$, $u^i(x^i_{-s}, e^i(s) + \frac{\tau^i_+(s)}{1+\delta} - \tau^i_-(s)) > u^i(x^i)$,[5] which implies in particular $e^i(s) + \frac{\tau^i_+(s)}{1+\delta} - \tau^i_-(s) \in \mathbb{R}^L_+$.

The assumptions **A.1**, **A.2**, **A.3** (1) and (2) are classical. **A.3** (3) and **A.4** (3) are constitutive of the model. Notice that **A.3** (1) implies that every asset has a strictly positive return in some state $s \in S$. Assumption **A.5** slightly differs from the usual gain to trade hypothesis. Unlike Tsomocos [19], it is done also at time $t = 0$.

We close this section with some properties necessarily verified by a monetary equilibrium under the assumptions **A.1**–**A.5**.

Proposition 2.1. *Let* $\left(p, q, \left(r_d(s), (r(s)\right)_{s\in S^*}, \rho, \left(\sigma_i\right)_{i\in I}, \left(\sigma_b\right)_{b\in B}\right)$ *be such that each* σ^i *is optimal in* $B^i\big(p, q, r, r_d, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}\big)$ *and each* σ^b *is optimal in* $B^b(q, r_d, r, \rho, (\sigma^i)_{i\in I})\}$. *Then,*

(a) $p \gg 0$, $q \gg 0$, $q_1 \geq \frac{1}{1+\rho}$ *and for all* $s \in S^*$, $0 \leq r_d(s) \leq r(s) \leq \rho$;

(b) *All end-of-period budget constraints are saturated. Cash-in-advance constraints at state* $s \in S^*$ *are saturated provided that* $r_d(s) > 0$.
This is true in particular at equilibrium where $r_d(s) > 0$ $\forall s \in S^*$.

(c) *At equilibrium, for each* $b \in B$, $\pi^b(s) > 0$ $\forall s \in S$; *consequently,* $v^i_b > 0$ *for each* $i \in I$ *and for every* $b \in B$.

Proof. That $p \gg 0$ follows by classical arguments from the strict monotonicity of utility functions u^i. Likewise, the financial market must not offer arbitrage opportunities. This implies $q \gg 0$. If we had $q_1 < \frac{1}{1+\rho}$, it would be possible for a bank b to increase simultaneously μ^b of one unit and z^b_1 of $\frac{1}{q_1(1+\rho)}$, still satisfying its budget constraints at time 0 but realizing at each $s \in S$ a positive profit of $\frac{1}{q_1(1+\rho)} - 1$, in contradiction with the optimality of σ^b. We next show that $r_d(s) \leq r(s)$, $\forall s \in S^*$. Assume that $r(0) < r_d(0)$. Observe that by increasing $\mu^i(0)$ by one unit and deposing $\frac{1}{1+r(0)}$ extra units, the cash-in advance constraint (1) is satisfied. Since $\frac{r(0)}{1+r(0)} < \frac{r_d(0)}{1+r(0)}$, the end-of-period constraint (2) holds with strict inequality. It follows that $x^i(0)$ can be increased, improving utility in contradiction to the optimality of σ^i. An analogous argument can be used to show that $r_d(s) \leq r(s)$, $\forall s \in S$. Observe that a symmetrical argument (that would conclude that $r_d(s) \geq r(s)$, $\forall s \in S^*$) cannot be used for banks which are submitted to the reserve requirements (10).

We subsequently show that $r(s) \leq \rho$, $\forall s \in S^*$. Assume that $\rho < r(0)$. Observe that by increasing μ^b by one unit and $m^b(0)$ by $\frac{1}{1+\rho}$ the cash-in-advance constraint (6) is satisfied and profit $\pi^b(0)$ increases by $\frac{1+r(0)}{1+\rho}$. Since $\frac{1+r(0)}{1+\rho} > 1$ the end-of-period constraint (9) at any state $s \in S$ holds with strict inequality, thus $\pi^b(s)$ can be increased in contradiction to the optimality of σ^b. Finally assume that $\rho < r(s)$ at some date-event $s \in S$. Increasing μ^b by one unit the cash-in-advance constraint at time zero is satisfied and profit $\pi^b(0)$ increases by $\frac{1}{1+\rho}$. This allows to increase $m^b(s)$ by $\frac{1}{1+\rho}$ without violating the cash-in-advance constraint at time 1. Since $\frac{1+r(s)}{1+\rho} > 1$, the end-of-period constraint at any state $s \in S$ holds with strict inequality, thus $\pi^b(s)$ can be increased in contradiction to the optimality of σ^b. This ends the proof of a.

[5] As usual, x^i_{-s} denotes the vector $\left(x^i(s)\right)_{s\in S^*\setminus\{s\}}$.

Saturation of budget constraints is obvious for the end-of-period budget constraints of consumers and banks. Assume now that for consumer i the constraint (1) is not saturated. If $r_d(0) > 0$, by increasing $\delta^i(0)$, it would be possible to increase $x^i(0)$, still satisfying (2), thus to improve the utility of consumer i, in contradiction with the optimality of σ^i. An analogue argument can be done in each state of nature at time 1.

For banks, if the constraint (6) is not saturated and if $r(0) \geq r_d(0) > 0$, then increasing $m^b(0)$ increases $\pi^b(0)$, thus the utility of bank b, in contradiction with the optimality of σ^b. If at state s, the constraint (8) is not saturated, bank b can increase $m^b(s)$), thus increase $\pi^b(s)$ still satisfying (9), in contradiction with the optimality of σ^b.

At equilibrium, the optimality of all equilibrium strategies σ^i and σ^b, joint to the market clearing relations (d) in in Definition 2.1, prevent any $r_d(s)$ to be zero.

To prove the last assertion, observe that the no-trade strategy which consists in doing nothing in the different asset and monetary markets belong to the budget set of each bank b and can afford to b the vector of profits $\widetilde{\pi}^b(0) = e^b_m(0)$, $\widetilde{\pi}^b(s) = e^b_m(0) + e^b_m(s)$ $\forall s \in S$. It follows from the strict monotonicity of the utility function u^b that $\pi^b(s) \geq e^b_m(0) + e^b_m(s) > 0$ $\forall s \in S$. Observe now that $v^b = \sum_{i\in I} v^i_b$ can be thought of as the equilibrium price of the profit shares in the ownership of firm b, an asset whose returns in each state $s \in S$ are equal to $\pi^b(s)$. As this market must not offer arbitrage opportunities to consumers, $v^i_b > 0$. ■

Proposition 2.2. *Let* $\left(p, q, \left(r_d(s), (r(s)\right)_{s\in S^*}, \rho, \left(\sigma_i\right)_{i\in I}, \left(\sigma_b\right)_{b\in B}\right)$ *be a monetary equilibrium. Then,*

(a) $\sum_{i\in I} \pi^i(0) + \sum_{b\in B} \pi^b(0) = M(0) + M^{CB}$;

(b) *If* $M^{CB} > 0$, *then for each* $s \in S$, $1 + \rho = \frac{M(0)+M^{CB}+M(s)}{M^{CB}} = 1 + \delta$.

Proof. a) Summing the end-of-period 0 constraints across consumers and banks, using the equilibrium market clearing conditions stated in Definition 2.1, and taking in account that the net revenue of State equals the net expenditure of agents on the asset market, we obtain:

$$\sum_{i\in I} \pi^i(0) + \sum_{b\in B} \pi^b(0) = M(0) + M^{CB}.$$

b) Summing the end-of-period 1 at state s constraints across consumers and banks, using the equilibrium market clearing conditions stated in Definition 2.1, and taking in account that the public debt is reimbursed at time 1, we obtain:

$$(1+\rho)M^{CB} = M(s) + \sum_{i\in I} \pi^i(0) + \sum_{b\in B} \pi^b(0).$$

It follows from Assumption **A.4** (3) and (4) and the definition of δ that

$$(1+\rho) = \frac{M^{CB} + M(0) + M(s)}{M^{CB}} = 1 + \delta$$

holds for each $s \in S$. ■

Our assumption on the constancy over $s \in S$ of the sum of the private monetary endowments of the agents is justified by this result. This is compatible with the assumption made in [14] for a model without intermediate banking system that consumers have no monetary endowment at time $t = 1$.

Observe that at equilibrium, each $x^i(s)$ satisfies $0 \leq x^i(s) \leq \sum_{i' \in I} x^{i'}(s) = e(s)$. Notice also that for each $i \in I$ and for each $b \in B$, $\pi^i(0) \leq M(0) + M^{CB}$ and $\pi^b(0) \leq M(0) + M^{CB}$. We have in addition:

Proposition 2.3. *The different monetary and portfolio quantities of a monetary equilibrium* $\big(p, (\sigma_i)_{i \in I}, (\sigma_b)_{b \in B}\big)$ *are bounded. More precisely,*

(a) *Without lost of generality,* $\delta^b = e^b_m(0)\ \forall b \in B$, *so that* $\sum \mu^b = (1+\delta)(\sum_b e^b_m(0) + M^{CB})$;

(b) *For each* b *and for each* $s \in S^*$, $d^b(s) \leq \frac{1}{k^1_s} e^b_m(0)$ *and* $m^b(s) \leq \frac{1}{k^2_s} e^b_m(s)$, *one of both constraints being binded at equilibrium. So that for each* $s \in S^*$,

$$0 \leq \sum_i \delta^i(s) \leq \frac{1}{k^1_s} \sum_b e^b_m(0) \quad \textit{and} \quad 0 \leq \sum_i \mu^i(s) \leq (1+\delta) \frac{1}{k^2_s} \sum_b e^b_m(s);$$

(c) *There exists* $A \in \mathbb{R}^J_{++}$ *such that* $\sum_i z^i_- + \sum_b z^b_- \leq A$;

(d) *Consequently,* $\sum_i z^i_+ + \sum_b z^b_+ \leq A + z^G$;

(e) *Each* v^i_b *is bounded;*

(f) *Finally, for each* $s \in S$, *each* $\pi^b(s)$ *is bounded.*

Proof. Assertion a. follows from market clearing.

It follows from the constraints (5) and the reserve and capital requirements (10) and (11) that for each $b \in B$, $0 \leq d^b(s) \leq \frac{1}{k_s} e^b_m(0)$ and $0 \leq m^b(s) \leq \frac{1}{k^2_s} e^b_m(s)$. It is easily seen that one of both constraints is binded at equilibrium. The remainder of Assertion b. follows from market clearing relations and the definition of Π.

Using for each i and for each b relations (3), (8), Assertion a. of the previous proposition and the short term credit and deposit market clearing, one sees that for every $s \in S$,

$$0 \leq R(s) \cdot \big(\sum_{i \in} z^i_- + \sum_{b \in B} z^b_-\big) \leq M(0) + M^{CB} + M(s).$$

Then, Assertion d. follows by classical arguments from Assumption **A.3** (1).

Assertion e. follows from the previous assertion and the feasibility condition $\sum_{i \in I} z^i_+ + \sum_{b \in B} z^b_+ = z^G + \sum_{i \in I} z^i_- + \sum_{b \in B} z^b_-$. More precisely, let $A \in \mathbb{R}^J_{++}$ be such that for all $i \in I$, $0 \leq z^i_- \leq \sum_i z^i_- + \sum_b z^b_- \leq A$, and for all $b \in B$, $0 \leq z^b_- \leq \sum_i z^i_- + \sum_b z^b_- \leq A$. Then, for all $i \in I$, $0 \leq z^i_+ \leq \sum_i z^i_+ + \sum_b z^b_+ \leq A + z^G$, and for all $b \in B$, $0 \leq z^b_+ \leq \sum_i z^i_+ + \sum_b z^b_+ \leq A + z^G$.

It then follows from the previous results and relations (1) that for each $i \in I$, $\sum_b v^i_b$ is bounded and from relations (9) that for each $b \in B$ and every $s \in S$, $\pi^b(s)$ is bounded. ■

Finally, the next proposition enlightens the role of Assumption **A.5**.

Proposition 2.4. *Let* $\big(p, (\sigma_i)_{i \in I}, (\sigma_b)_{b \in B}\big)$ *be a monetary equilibrium. At each state* $s \in S^*$ *there is some short term monetary activity.*

Proof. Let us first assume that for some $s \in S$, $\sum_i \mu^i(s) = \sum_i \delta^i(s) = 0$. A glimpse at (binded) constraints (3) and (4) shows that for each i, $p(s) \cdot \big(x^i(s) - e^i(s)\big)^- = 0$, which, in view of the strict positivity of $p(s)$ and the market clearing equation, implies $x^i = e^i$. It now follows from the optimality of banks that $r_d(s) = r(s)$. Indeed, if we had $r_d(s) < r(s)$, some bank b, with some $d'^b(s) > 0$ and $m'^b = \frac{1}{1+r_d(s))} d'^b$, could satisfy constraints (8), (9), (10), (11) and increase its profit $\pi^b(s)$, in contradiction with the optimality of b.

Then, let $\big(\tau^i(s),\ i \in I\big)$ be the transfers referred to in Assumption **A.5**. For each i, there exist some $\delta'^i(s), \mu'^i(s)$ such that $p(s)\cdot\frac{\tau^i_+(s)}{1+\delta}+\delta'^i(s)-\frac{\mu'^i(s)}{1+r(s)}=0$ and $p(s)\cdot\big(\frac{\tau^i_+(s)}{1+\delta}-\tau^i(s)\big)-r_d(s)\delta'^i(s)+\frac{r(s)}{1+r(s)}\mu'^i(s)>0$. Combining the two previous relations, using $r_d(s)=r(s)$, recalling that, in view of propositions 2.1 and 2.2, $\frac{1+r(s)}{1+\delta}\le 1$, we get $p(s)\cdot\tau^i(s)>0$ for all $i\in I$ and, summing on $i\in I$, $0=p(s)\cdot\sum_{i\in I}\tau^i(s)>0$, a contradiction.

Let us now assume that $\sum_i \mu^i(0)=\sum_i\delta^i(0)=0$. Using the (binded) constraints (1) and (2), we first get for each $i\in I$, $\pi^i(0)=p(0)\cdot\big(x^i(0)-e^i(0)\big)^-+q\cdot z^i_-$. We first claim that $\big(x^i(0)-e^i(0)\big)^-=0$. Indeed, if not, by increasing $q\cdot z^i_-$, one can decrease $p(0)\cdot\big(x^i(0)-e^i(0)\big)^-$, thus increase the utility of consumer i, without changing $\pi^i(0)$. Since, by Proposition 2.1, $\sum_b\theta^i_b\pi^b(s)>0\ \forall s\in S$, Consumer i can still verify his budget constraints (3) and (4) via an appropriate choice in each $s\in S$ of $\mu^i(s)$ and $\delta^i(s)$. From $\big(x^i(0)-e^i(0)\big)^-=0$ and market clearing at time $t=0$, we deduce $x^i(0)=e^i(0)\ \forall i\in I$.

Then, using the Gain to trade hypothesis **A.5** for time $t=0$, the rest of the proof goes as previously for $s\in S$. ■

3. Existence of monetary equilibrium

The equilibrium existence will be proved first for a compact economy $\mathcal{E}^c$, defined using the previous results. It will be the consequence of a fixed point argument in this economy with modified budget sets for banks and consumers. The definition of modified budget sets using state prices of money at time 0 and in the different states of the world is inspired by Gourdel–Triki [14], the fixed point argument follows Florenzano [9]. We will use the following fixed point result which extends Kakutani [16] and Gale and Mas-Colell [10, 11]:

Lemma 3.1 (Gourdel [13])**.** *Let for each $i=1,\cdots,m$, X^i be a convex and compact subset of some Euclidean vector space, $X=\prod_{i=1}^m X^i$ and $\varphi^i\colon X\to X^i$ be a convex valued correspondence which is either lower semicontinuous or upper semicontinuous with closed values. Then, there exists $\overline{x}=(\overline{x}^i)_{i=1}^m\in X$ such that for each i, either $\overline{x}^i\in\varphi^i(\overline{x})$ or $\varphi^i(\overline{x})=\emptyset$.*

3.1. Truncating the economy and modifying budget sets.

If we set $\pi=(p,q,\frac{1}{1+r_d},\frac{1}{1+r},\frac{1}{1+\rho})$, in view of the previous results, we will restrict the prices to belong to the convex and compact set:

$$\Pi=\left\{\pi\in\mathbb{R}^{LS^*}_+\times\mathbb{R}^J_+\times\mathbb{R}^{S^*}_+\times\mathbb{R}^{S^*}_+\times\mathbb{R}_+\ \middle|\ \begin{array}{l}\|p(0)\|_1+\|q\|_1=1\\ \|p(s)\|_1=1,\ \forall s\in S\\ \frac{1}{1+\delta}\le\frac{1}{1+r_d(s)}\le 1,\ \forall s\in S^*\\ \frac{1}{1+\delta}\le\frac{1}{1+r(s)}\le 1,\ \forall s\in S^*\\ \frac{1}{1+\delta}\le\frac{1}{1+\rho}\le 1\end{array}\right\}.$$

We next define the strategy sets Σ^i_c and Σ^i_c of $\mathcal{E}^c$ by the following inequalities on real and monetary variables:

- $\sum_b\delta^b=\sum_b e^b_m(0)$;
- $\sum_b e^b_m(0)+M^{CB}\le\sum_b\mu^b\le(1+\delta)\big(\sum_b e^b_m(0)+M^{CB}\big)$;

and for all $s\in S^*$,

- $\frac{1}{1+\delta}\frac{1}{k^1_s}\sum_b e^b_m(0)\le\sum_b d^b(s)\le\frac{1}{k^1_s}\sum_b e^b_m(0)$;
- $\frac{1}{1+\delta}\frac{1}{k^1_s}\sum_b e^b_m(0)\le\sum_i\delta^i(s)\le\frac{1}{k^1_s}\sum_b e^b_m(0)$;
- $\frac{1}{1+\delta}\frac{1}{k^2_s}\sum_b e^b_m(s)\le\sum_b m^b(s)\le\frac{1}{k^2_s}\sum_b e^b_m(s)$;

- $\frac{1}{1+\delta}\frac{1}{k_s^2}\sum_b e_m^b(s) \le \sum_i \mu^i(s) \le \frac{1}{k_s^2}\sum_b e_m^b(s)$.

For consumption vectors, we assume that $\sum_i x^i \le 2e$.
For portfolios, we assume that $\sum_i z_-^i + \sum_b z_-^b \le 2A$ and $\sum_i z_+^i + \sum_b z_+^b \le 3A + 2z^G$.
On the other hand, let $a = \sup\{p(0)\cdot e(0) + q\cdot A\colon \|p\|_1 + \|q\|_1 = 1\}$. We set

$$\sum_i \pi^i(0) + \sum_b \pi^b(0) \le 2\ (1+\delta)a + 2\ \big(M(0) + M^{CB}\big)$$

and assume that for each i,

$$\sum_b \upsilon_b^i \le 2\big[\sum_i e_m^i(0) + (1+\delta)\frac{1}{k_0^2}\sum_b e_m^b(0)\big].$$

For profits at each $s \in S$, we assume

$$\sum_b \pi^b(s) \le M(0) + M(s) + M^{CB} + R(s)\cdot(A + z^G) + \delta(1+\delta)\frac{1}{k_s^2}\sum_b e_m^b(s).$$

The truncated economy is

$$\mathcal{E}^c = \Big((\Sigma_c^i, u^i, e^i, e_m^i)_{i\in I}, (\Sigma_c^b, u^b, e_m^b)_{b\in B}, R, (k_s^1, k_s^2)_{s\in S^*}, z_1^G, M^{CB}\Big).$$

Let us now set

$$M = \mathrm{co}\left\{\mu \in \mathbb{R}_+^{S^*} \;\middle|\; \begin{array}{l} \exists\big((\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big) \in \prod_{i\in I}\Sigma_c^i \times \prod_{b\in B}\Sigma_c^b,\ \exists \pi \in \Pi \ \text{ such that} \\ \mu_0[M(0) + M^{CB}] = \sum_i p(0)\cdot\big(x^i(0) - e^i(0)\big)^- + \\ \hfill q\cdot\big(\sum_i z_-^i + \sum_b z_-^b + z^G\big) \\ \mu_s[M(s) + \sum_i \pi^i(0) + \sum_b \pi^b(0)] = \sum_i p(s)\cdot\big(x^i(s) - e^i(s)\big)^- \end{array}\right\}.$$

By definition, M is a convex subset of $\mathbb{R}_+^{S^*}$. As easily verified using the assumptions **A.3**(1) and **A.4**(3), M is also compact. The economic interpretation of the equalities defining M is that, at each state $s \in S^*$, the actualized value of the total money in the hands of agents should equilibrate the anticipation of their revenues at the end of the period.

We define in $\mathcal{E}^c$ the budget sets of consumers $B^i\big(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu\big)$, replacing the constraints (1), (2) at time 0 by

$$p(0)\cdot\big(x^i(0) - e^i(0)\big)^+ + q\cdot z_+^i + \delta^i(0) - \frac{\mu^i(0)}{1+r(0)} \le \mu_0[e_m^i(0) - \sum_{b\in B}\upsilon_b^i] \tag{12}$$

$$p(0)\cdot\big(x^i(0) - e^i(0)\big) + q\cdot z^i - r_d(0)\delta^i(0) + \frac{r(0)}{1+r(0)}\mu^i(0) \le \mu_0[e_m^i(0) - \sum_{b\in B}\upsilon_b^i - \pi^i(0)] \tag{13}$$

and replacing the constraints (3) and (4) at each state s of time $t = 1$ by

$$p(s)\cdot\big(x^i(s) - e^i(s)\big)^+ + \delta^i(s) - \frac{\mu^i(s)}{1+r(s)} \le \mu_s\big[e_m^i(s) + \pi^i(0) - R(s)\cdot z_-^i\big] \tag{14}$$

$$p(s)\cdot\big(x^i(s) - e^i(s)\big) - r_d(s)\delta^i(s) + \frac{r(s)}{1+r(s)}\mu^i(s) \le \mu_s\big[e_m^i(s) + \pi^i(0) + R(s)\cdot z^i + \sum_{b\in B}\theta_b^i\pi^b(s)\big]. \tag{15}$$

For banks, we define in $\mathcal{E}_c$ the budget sets of banks $B^b\Big(\pi, (\sigma^i)_{i\in I}, \mu\Big)$, replacing at time 0 the constraints (6), (7) by

$$q \cdot z^b_+ + m^b(0) - \frac{d^b(0)}{1+r_d(0)} \leq \mu_0\big[e^b_m(0) + \frac{\mu^b}{1+\rho} - \delta^b + \sum_{i\in I} \upsilon^i_b\big] \tag{16}$$

$$q \cdot z^b + \frac{r_d(0)}{1+r_d(0)} d^b(0) - r(0)m^b(0) \leq \mu_0\big[e^b_m(0) - \pi^b(0) + \frac{\mu^b}{1+\rho} - \delta^b + \sum_{i\in I} \upsilon^i_b\big] \tag{17}$$

and replacing at each state s of period 1 the constraints (8) and (9) by

$$m^b(s) - \frac{d^b(s)}{1+r_d(s)} \leq \mu_s\big[e^b_m(s) + \pi^b(0) - R(s) \cdot z^b_-\big] \tag{18}$$

$$\frac{r_d(s)}{1+r_d(s)} d^b(s) - r(s)m^b(s) \leq \mu_s\big[e^b_m(s) + \pi^b(0) - \pi^b(s) + (1+\rho)\delta^b - \mu^b + R(s) \cdot z^b\big]. \tag{19}$$

The reserve and capital requirements (10) and (11) are replaced by

$$k^1_s d^b(s) \leq \mu_s \delta^b \ \forall s \in S^*; \tag{20}$$

$$k^2_s m^b(s) \leq \mu_s e^b_m(s) \ \forall s \in S^*. \tag{21}$$

For each $i \in I$ (resp. $b \in B$), for each $(\pi, (\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}, \mu) \in \Pi \times \prod_{i\in I} \Sigma^i_c \times \prod_{b\in B} \Sigma^b_c \times M$ let us denote by $B'^i\big(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu\big)$ (resp. $B'^b\big(\pi, (\sigma_b)_{i\in I}, \mu\big)$) the sets deduced from $B^i\big(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu\big)$ (resp. $B^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big)$) replacing all budget inequalities by strict inequalities. Let us also denote $\underline{\sigma}^i = (e^i, 0, 0, 0, 0, 0)$ and $\underline{\sigma}^b = (0, 0, 0, 0, 0, 0)$ the no-trade strategies for consumers and banks and define the correspondences:

$$B''^i\big(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu\big) = \begin{cases} \{\underline{\sigma}^i\} & \text{if } B'^i\big(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu\big) = \emptyset \\ B'^i\Big(\pi, \big(\pi^b(s)_{s\in S}\big)_{b\in B}, \mu\Big) & \text{if } B'^i\big(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu\big) \neq \emptyset \end{cases}$$

$$B''^b\Big(\pi, (\sigma^i)_{i\in I}, \mu\Big) = \begin{cases} \{\underline{\sigma}^b\} & \text{if } B'^b\Big(\pi, (\sigma^i)_{i\in I}, \mu\Big) = \emptyset \\ B'^b\Big(\pi, (\sigma^i)_{i\in I}, \mu\Big) & \text{if } B'^b\Big(\pi, (\sigma^i)_{i\in I}, \mu\Big) \neq \emptyset \end{cases}.$$

Proposition 3.1. *In $\mathcal{E}^c$, for each $i \in I$ and for all $(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu) \in \Pi \times \prod_{b\in B} \Sigma^b_c \times \prod_{i'\neq i} \Sigma^{i'}_c \times M$, $\underline{\sigma}^i \in B^i(\pi, (\sigma^b)_{b\in B}, (\sigma^{i'})_{i'\neq i}, \mu)$; for each $b \in B$ and for all $(\pi, (\sigma^i)_{i\in I}, \mu) \in \Pi \times \Sigma^i_c \times M$, $\underline{\sigma}^b \in B^b(\pi, (\sigma^i)_{i\in I}, \mu)$. Moreover,*

(a) *For each $i \in I$, B^i is upper semicontinuous on $\Pi \times \prod_{b\in B} \Sigma^b_c \times \prod_{i'\neq i} \Sigma^{i'}_c \times M$ with closed convex values; B'^i has an open graph in $\Pi \times \prod_{b\in B} \Sigma^b_c \times \prod_{i'\neq i} \Sigma^{i'}_c \times M \times \Sigma^i_c$ and B''^i is lower semicontinuous.*

(b) *For each $b \in B$, B^b is upper semicontinuous on $\Pi \times \prod_{i\in I} \Sigma^i_c \times M$ with closed convex values; B'^b has an open graph in $\Pi \times \prod_{i\in I} \Sigma^i_c \times M \times \Sigma^b_c$ and B''^b is lower semicontinuous.*

Proof. For a proof of the lower semicontinuity of B''^i or of B''^b, use a similar argument to the one used in Florenzano[9], claim 7.2 p.23. The proofs of the other continuity properties are standard. ■

3.2. **The fixed point argument.** The equilibrium existence problem in $\mathcal{E}^c$ with the modified budget sets is clearly a problem of simultaneous optimization that we will solve by finding a fixed point for the product of the correspondences that we now define:

$$\varphi^M\big(\pi, (\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big) =$$
$$\left\{\mu \in M \;\middle|\; \begin{array}{l} \mu_0[M(0) + M^{CB}] = \sum_i p(0)\cdot\big(x^i(0) - e^i(0)\big)^- \\ \qquad\qquad\qquad +q\cdot\big(\sum_i z^i_- + \sum_b z^b_- + z^G\big) \\ \mu_s[M(s) + \sum_i \pi^i(0) + \sum_b \pi^b(0)] = \sum_i p(s)\cdot\big(x^i(s) - e^i(s)\big)^- \end{array}\right\}$$

$$\varphi^0\big(\pi, (\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big) =$$
$$\left\{\widetilde{\pi} \in \Pi \;\middle|\; \begin{array}{l} \widetilde{p}\cdot\sum_i(x^i - e^i) + \widetilde{q}\cdot\big(\sum_i z^i + \sum_b z^b - z^G_1\big) > p\cdot\sum_i(x^i - e^i) + q\cdot\big(\sum_i z^i + \sum_b z^b - z^G\big) \\ \frac{1}{1+\widetilde{r}_d(s)} = \max\{\frac{1}{1+\delta}, \min\{1, \frac{\sum_i \delta^i(s)}{\sum_b d^b(s)}\}\}\ \forall s \in S^* \\ \frac{1}{1+\widetilde{r}(s)} = \max\{\frac{1}{1+\delta}, \min\{1, \frac{\sum_b m^b(s)}{\sum_i \mu^i(s)}\}\}\ \forall s \in S^* \\ \frac{1}{1+\widetilde{\rho}} = \max\{\frac{1}{1+\delta}, \min\{1, \{\frac{\sum_b \delta^b + M^{CB}}{\sum_b \mu^b}\}\} \end{array}\right\}$$

and for each $i \in I$ and for each $b \in B$, the following reaction correspondences:

$$\varphi^i\big(\sigma^i, \pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) = \begin{cases} B''^i\big(\pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) & \text{if } \sigma^i \notin B^i\big(\pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) \\ B'^i\big(\pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) \cap P^i(\sigma^i) & \text{if } \sigma^i \in B^i\big(\pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) \end{cases}$$

$$\varphi^b\big(\sigma^b, \pi, (\sigma^i)_{i\in I}, \mu\big) = \begin{cases} B''^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big) & \text{if } \sigma^b \notin B^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big) \\ B'^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big) \cap P^b(\sigma^b) & \text{if } \sigma^b \in B^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big) \end{cases}$$

where $P^i(\sigma^i) = \{\widetilde{\sigma}^i \mid u^i(\widetilde{x}^i) > u^i(x^i)\}$, $P^b(\sigma^b) = \{\widetilde{\sigma}^b \mid u^b(\widetilde{\pi}^b) > u^b(\pi^b)\}$, and $\sigma_{-i} = \prod_{i'\neq i}\sigma^{i'}$.

Finally,

$$\Phi\colon M \times \Pi \times \prod_{i\in I}\Sigma^i_c \times \prod_{b\in B}\Sigma^b_c \to M \times \Pi \times \prod_{i\in I}\Sigma^i_c \times \prod_{b\in B}\Sigma^b_c$$

is defined by:

$$\Phi\big(\mu, \pi, (\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big) = \varphi^M\big(\pi, (\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big) \times \varphi^0\big((\sigma^i)_{i\in I}, (\sigma^b)_{b\in B}\big)$$
$$\times \prod_{i\in I}\varphi^i\big(\sigma^i, \pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) \times \prod_{b\in B}\varphi^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big).$$

Notice that φ^0 (actually defined by a product of correspondences) describes the behavior of an hypothetic auctioneer who sets not only the commodity and asset prices but also the different interest rates in reaction to the market actions of the agents. The definition of φ^M is specific of our fixed point argument.

Under the assumptions **A.3** and **A.4**, φ^M is an upper semicontinuous correspondence. The first correspondence defining φ^0 has an open graph. The following ones are, by construction, continuous functions. Coming now to the definition of φ^i and φ^b, we remark that for each i, $\Big(B'^i\big(\pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big) \cap P^i(\sigma^i)\Big) \subset B''^i\big(\pi, (\sigma^b)_{b\in B}, \sigma_{-i}, \mu\big)$, and for each b, $\Big(B'^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big) \cap P^b(\sigma^b)\Big) \subset B''^b\big(\pi, (\sigma^i)_{i\in I}, \mu\big)$. It then follows from Proposition 3.1, the continuity of utility functions and their definition that each φ^i (resp. each φ^b) is a lower semicontinuous correspondence with convex values.

Applying Lemma 3.1, one gets immediately the following "fixed point" result:

Proposition 3.2. *There exists* $\big(\overline{\mu}, \overline{\pi}, (\overline{\sigma}^i)_{i\in I}, (\overline{\sigma}^b)_{b\in B}\big) \in M \times \Pi \times \Sigma_c^i \times \Sigma_c^b$ *such that:*

(a) $\overline{\mu}_0[M(0) + M^{CB}] = \sum_i \overline{p}(0) \cdot \big(\overline{x}^i(0) - e^i(0)\big)^- + \overline{q} \cdot \big(\sum_i \overline{z}_-^i + \sum_b \overline{z}_-^b + z^G\big)$,
$\forall s \in S,\ \overline{\mu}_s[M(s) + \sum_i \overline{\pi}^i(0) + \sum_b \overline{\pi}^b(0)] = \sum_i \overline{p}(s) \cdot \big(\overline{x}^i(s) - e^i(s)\big)^-$,

(b) $(p - \overline{p}) \cdot \sum_i (\overline{x}^i - e^i) + (q - \overline{q}) \cdot (\sum_i \overline{z}^i + \sum_b \overline{z}^b - z^G) \le 0$
for all $(p, q) \ge 0$ *such that* $\|p(0)\|_1 + \|q\|_1 = 1,\ \|p(s)\|_1 = 1,\ \forall s \in S$,

(c) $\forall s \in S^*,\ \frac{1}{1+\overline{r}_d(s)} = \max\big\{\frac{1}{1+\delta}, \min\{1, \frac{\sum_i \overline{\delta}^i(s)}{\sum_b \overline{d}^b(s)}\}\big\}$

(d) $\forall s \in S^*,\ \frac{1}{1+\overline{r}(s)} = \max\big\{\frac{1}{1+\delta}, \min\{1, \frac{\sum_b \overline{m}^b(s)}{\sum_i \overline{\mu}^i(s)}\big\}$

(e) $\frac{1}{1+\overline{\rho}} = \max\big\{\frac{1}{1+\delta}, \min\{1, \frac{\sum_b \overline{\delta}^b + M^{CB}}{\sum_b \overline{\mu}^b}\}\big\}$

(f) *for each* $i \in I$, $\overline{\sigma}^i \in B^i\big(\overline{\pi}, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big)$ *and* $B'^i\big(\pi, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big) \cap P^i(\overline{\sigma}^i) = \emptyset$,

(g) *for each* $b \in B$, $\overline{\sigma}^b \in B^b\big(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu}\big)$ *and* $B'^b\big(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu}\big) \cap P^b(\overline{\sigma}^b) = \emptyset$.

Remark 3.2. Since each $\overline{\sigma}^b$ is in $B^b\big(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu}\big)$, it follows from the truncation and from the capital and reserve requirements in the modified budget sets that $\overline{\mu}_s \ge \frac{1}{1+\delta}\ \forall s \in S^*$.

We are now in conditions of proving the main result of this section.

Proposition 3.3. *Under the assumptions* **A.1** – **A.5**, *the economy* $\mathcal{E}^c$ *has a monetary equilibrium. Consequently, under the same conditions,* $\mathcal{E}$ *has a monetary equilibrium.*

Proof. Let $\big(\overline{\mu}, \overline{\pi}, (\overline{\sigma}^i)_{i\in I}, (\overline{\sigma}^b)_{b\in B}\big)$ satisfy the assertions of the previous proposition. The proof of the first assertion of Proposition 3.3 will be done in a series of claims establishing properties of the "equilibrium point" $\big(\overline{\mu}, \overline{\pi}, (\overline{\sigma}^i)_{i\in I}, (\overline{\sigma}^b)_{b\in B}\big)$.

Claim 3.3. (a) *One has*

$$\frac{\sum_b \overline{\delta}^b + M^{CB}}{\sum_b \overline{\mu}^b} \ge \frac{1}{1+\delta} \tag{22}$$

and for every $s \in S^*$,

$$\frac{\sum_i \overline{\delta}^i(s)}{\sum_b \overline{d}^b(s)} \ge \frac{1}{1+\delta} \quad ; \quad \frac{\sum_b m^b(s)}{\sum_i \mu^i(s)} \ge \frac{1}{1+\delta}. \tag{23}$$

(b) *Consequently,*

$$\frac{1}{1+\overline{\rho}} \sum_b \overline{\mu}^b \le \sum_b \overline{\delta}^b + M^{CB} \tag{24}$$

and for every $s \in S^*$,

$$\frac{1}{1+\overline{r}_d(s)} \sum_b \overline{d}^b(s) \le \sum_i \overline{\delta}^i(s) \quad ; \quad \frac{1}{1+\overline{r}(s)} \sum_i \overline{\mu}^i(s) \le \sum_b \overline{m}^b(s). \tag{25}$$

a. is proved using the bounds which define Σ_c^i and Σ_c^b. b. follows from a. and the relations giving the short term and long term interest rates in Proposition 3.2.

Claim 3.4. *a. At time* 0, $\sum_{i\in I}\big(\overline{x}^i(0) - e^i(0)\big) \le 0$ *and* $\sum_{i\in I} \overline{z}^i + \sum_{b\in B} \overline{z}^b \le z^G$.
b. At each state s *of time* 1, $\sum_{i\in I}\big(\overline{x}^i(s) - e^i(s)\big) \le 0$.

To prove a., let us remark that it follows easily from $\overline{\pi} \in \varphi^0\big((\overline{\sigma}^i)_{i\in I}, (\overline{\sigma}^b)_{b\in B}\big)$ that for each $\big(p(0), q\big) \geq 0$ such that $\|p(0)\|_1 + \|q\|_1 = 1$,

$$p(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big)+q\cdot(\sum_i\overline{z}^i+\sum_b\overline{z}^b-z^G) \leq \overline{p}(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big)+\overline{q}\cdot(\sum_i\overline{z}^i+\sum_b\overline{z}^b-z^G).$$

On the other hand, each $\overline{\sigma}^i \in B^i\big(\overline{\pi}, ((\overline{\sigma}_b)_{b\in B}, \overline{\sigma}_-i, \overline{\mu}\big)$ and $\overline{\sigma}^b \in B^b(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu})$. Adding the cash in advance constraints (12) and (16) at time 0 of consumers and banks, we get:

$$\overline{p}(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big)^+ + \overline{q}\cdot\big(\sum_i\overline{z}^i_+ + \sum_b\overline{z}^b_+\big) + \big(\sum_i\overline{\delta}^i(0) - \frac{1}{1+\overline{r}_d(0)}\sum_b\overline{d}^b(0)\big)$$
$$+\big(\sum\overline{m}^b(0) - \frac{1}{1+\overline{r}(0)}\sum_i\overline{\mu}^i(0)\big) \leq \overline{\mu}_0\big[M(0) + (\frac{1}{1+\overline{\rho}}\sum_b\overline{\mu}^b - \sum_b\overline{\delta}^b)\big].$$

Taking in account the relations proved in b. of the previous claim, we deduce:

$$\overline{p}(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big)^+ + \overline{q}\cdot\big(\sum_i\overline{z}^i_+ + \sum_b\overline{z}^b_+\big) \leq \overline{\mu}_0\big[M(0) + M^{CB}\big]$$

From Proposition 3.2, we have also:

$$\overline{\mu}_0[M(0) + M^{CB}] = \overline{p}(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big)^- + \overline{q}\cdot\big(\sum_i\overline{z}^i_- + \sum_b\overline{z}^b_- + z^G\big).$$

Combining the previous relations, we get:

$$\overline{p}(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big) + \overline{q}\cdot\big(\sum_i\overline{z}^i + \sum_b\overline{z}^b - z^G\big) \leq 0.$$

Since for each $\big(p(0), q\big) \in \mathbb{R}^L_+ \times \mathbb{R}^J_+$ such that $\|p(0)\|_1 + \|q\|_1 = 1$,

$$p(0)\cdot\sum_i\big(\overline{x}^i(0)-e^i(0)\big) + q\cdot(\sum_i\overline{z}^i + \sum_b\overline{z}^b - z^G) \leq 0,$$

the part a. of our claim is proved.

For each $s \in S$, it follows from $\overline{\pi} \in \varphi^0\big((\overline{\sigma}^i)_{i\in I}, (\overline{\sigma}^b)_{b\in B}\big)$ that for each $p(s) \geq 0$ such that $\|p(s)\|_1 = 1$,

$$p(s)\cdot\sum_i\big(\overline{x}^i(s)-e^i(s)\big) \leq \overline{p}(s)\cdot\sum_i\big(\overline{x}^i(s)-e^i(s)\big).$$

Since each $\overline{\sigma}^i \in B^i\big(\overline{\pi}, ((\overline{\sigma}_b)_{b\in B}, \overline{\sigma}_-i, \overline{\mu}\big)$ and $\overline{\sigma}^b \in B^b(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu})$, adding the cash in advance constraints (14) and (18) at state s of consumers and banks, we get for the "fixed point" of Φ:

$$\overline{p}(s)\cdot\sum_i\big(\overline{x}^i(s)-e^i(s)\big)^+ + \big(\sum_i\overline{\delta}^i(s) - \frac{1}{1+\overline{r}_d(s)}\sum_b\overline{d}^b(s)\big) + \big(\sum\overline{m}^b(s) - \frac{1}{1+\overline{r}(s)}\sum_i\overline{\mu}^i(s)\big)$$
$$\leq \overline{\mu}_s\big[M(s) + \sum_i\overline{\pi}^i(0) + \sum_b\overline{\pi}^b(0) - R(s)\cdot\big(\sum_i\overline{z}^i_- + \sum_b\overline{z}^b_-\big)\big] \leq \overline{\mu}_s\big[M(s) + \sum_i\overline{\pi}^i(0) + \sum_b\overline{\pi}^b(0)\big].$$

From this and relations proved in b. of the previous claim, we get:

$$\overline{p}(s)\cdot\sum_i\big(\overline{x}^i(s)-e^i(s)\big)^+ \leq \overline{\mu}_s\big[M(s) + \sum_i\overline{\pi}^i(0) + \sum_b\overline{\pi}^b(0)\big].$$

From Proposition 3.2, we have also:

$$\overline{\mu}_s[M(s) + \sum_i\overline{\pi}^i(0) + \sum_b\overline{\pi}^b(0)] = \sum_i\overline{p}(s)\cdot\big(\overline{x}^i(s)-e^i(s)\big)^- \tag{26}$$

We thus deduce:

$$\overline{p}(s) \cdot \sum_i \left(\overline{x}^i(s) - e^i(s)\right)^+ \leq \sum_i \overline{p}(s) \cdot \left(\overline{x}^i(s) - e^i(s)\right)^-,$$

that is, $\overline{p}(s) \cdot \sum_i \big(\overline{x}^i(s) - e^i(s)\big) \leq 0)$. Since for each $p(s) \in \mathbb{R}^L_+$ such that $\|p(s)\|_1 = 1$,

$$p(s) \cdot \sum_i \big(\overline{x}^i(s) - e^i(s)\big) \leq \overline{p}(s) \cdot \sum_i \big(\overline{x}^i(s) - e^i(s)\big) \leq 0,$$

the part b. of our claim is proved.

Remark 3.5. One will keep in mind that it follows from the previous claim that $\sum_i \overline{x}^i \ll 2e$ and that $\sum_i \overline{z}^i_+ + \sum_b \overline{z}^b_+ \ll 3A + 2z^G$.

Claim 3.6. *For some $i \in I$, $\overline{\sigma}^i \in \operatorname{argmax}\left\{u^i(x^i) \colon \sigma^i \in B^i\big(\overline{\pi}, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big)\right\}$.*

As it follows from a classical argument, to prove the claim, it suffices to verify that for some $i \in I$, $B'^i\big(\overline{\pi}, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big) \neq \emptyset$. To see that, for each $i \in I$ and in each $s \in S$, one can first choose $\mu^i(s)$ such than $0 < \mu^i(s) < \overline{p}(s) \cdot e^i(s)$ and z^i such that $\mu_s R(s) \cdot z^i_- < \frac{\mu^i(s)}{1+\overline{r}(s)}$ $\forall s \in S$ and $\overline{q} \cdot z^i_- > 0$ if $\overline{q} \neq 0$. At state 0, $\mu^i(0)$ must be chosen such that $\mu^i(0) > 0$ and $\frac{\overline{r}(0)}{1+\overline{r}(0)}\mu^i(0) < \overline{p}(0) \cdot e^i(0) + \overline{q} \cdot z^i_-$. This is possible for every $i \in I$ if $\overline{q} \neq 0$. and at least for one $i \in I$ if $\overline{q} = 0$, according to the normalization of prices at period 0 and the survival assumption in **A.4**.

Claim 3.7. *Consequently, $\overline{p} \gg 0$, $\overline{q} \gg 0$, and, for each $i \in I$,*

$$\overline{\sigma}^i \in \operatorname{argmax}\left\{u^i(x^i) \colon \sigma^i \in B^i\big(\overline{\pi}, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big)\right\}.$$

In view of Assumption **A.3** (1), each asset has a nonzero return in some state $s \in S$. In view of Remark 3.5, that $\overline{p} \gg 0$ and $\overline{q} \gg 0$ follow from the previous claim.

The proof that $B'^i\big(\overline{\pi}, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big) \neq \emptyset$ is as in Claim 3.6, even simpler.

Claim 3.8. *For each $b \in B$, $\overline{\sigma}^b \in \operatorname{argmax}\left\{u^b(\pi^b) \colon \sigma^b \in B^b\big(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu}\big)\right\}$.*

For each $b \in B$, the optimality of $\overline{\sigma}^b$ implies that $\overline{\pi}^b(s) > 0$ for all $s \in S$ and thus that $\sum_{i\in I} \overline{\theta}^i_b = 1$.

Since $\overline{q}_1 > 0$, using the assumption **A.4** (2), for each $b \in B$, one can choose z^b_+ such that $\overline{q} \cdot z^b_+ < \overline{\mu}_0 e^b_m(0)$ and for each $s \in S$, $d^b(s) > 0$ such that $\frac{\overline{r}_d(s)}{1+\overline{r}_d(s)} d^b(s) < \overline{\mu}_s R(s) \cdot z^b_+$. This shows that $B'^b\Big(\overline{\pi}, (\overline{v}^i_b)_{i\in I}, \overline{\mu}\Big) \neq \emptyset$, thus that $\overline{\sigma}^b$ is optimal in $B^b\Big(\overline{\pi}, (\overline{v}^i_b)_{i\in I}, \overline{\mu}\Big)$.

To prove the last assertion, observe that the no-trade strategy which consists in doing nothing in the different asset and monetary markets can afford to the bank b the vector of profits $\widetilde{\pi}^b(0) = e^b_m(0)$, $\widetilde{\pi}^b(s) = e^b_m(0) + e^b_m(s)$ $\forall s \in S$. It follows from the strict monotonicity of the utility function u^b that $\overline{\pi}^b(s) \geq e^b_m(0) + e^b_m(s) > 0$ $\forall s \in S$. Observe now that $\overline{v}^b = \sum_{i\in I} \overline{v}^i_b$ can be thought of as the equilibrium price of the profit shares in the ownership of firm b, an asset whose returns in each state $s \in S$ are equal to $\overline{\pi}^b(s)$. As this market must not offer arbitrage opportunities to consumers, $\overline{v}^b = \sum_{i\in I} \overline{v}^i_b > 0$. It thus follows that $\sum_{i\in I} \overline{\theta}^i_b = 1$.

Claim 3.9. *The long term monetary market and all short term monetary markets clear.*

For the long term monetary market, market clearing simply follows from the bounds on $\sum_b \overline{\mu}^b$.

Now, by definition, if $r_d(s) > 0$, market clearing follows. Recalling that, in view of Remark 3.5, $\sum_i \overline{x}^i(s) \ll 2e$, if $r_d(s) = 0$, it follows from the optimality of each $\overline{\sigma}^i$ that $\sum_i \overline{\delta}^i(s) =$

$\frac{1}{1+\delta}\frac{1}{k_s^1}\sum_b e_m^b(0)$. We then have $\frac{\sum_i \overline{\delta}^i(s)}{\sum_b \overline{d}^b(s)} \leq 1$, which shows market clearing for the short term market of deposit.

The proof of market clearing for the short term market for borrowing and lending is similar.

Claim 3.10. *One has actually at the "fixed point":*

- $\sum_b e_m^b(0) + M^{CB} < \sum_b \overline{\mu}^b < (1+\delta)\big(\sum_b e_m^b(0) + M^{CB}\big)$;

and for all $s \in S^*$,

- $\frac{1}{1+\delta}\frac{1}{k_s^1}\sum_b e_m^b(0) < \sum_b \overline{d}^b(s) < \frac{1}{k_s^1}\sum_b e_m^b(0)$;
- $\frac{1}{1+\delta}\frac{1}{k_s^1}\sum_b e_m^b(0) < \sum_i \overline{\delta}^i(s) < \frac{1}{k_s^1}\sum_b e_m^b(0)$;
- $\frac{1}{1+\delta}\frac{1}{k_s^2}\sum_b e_m^b(s) < \sum_b \overline{m}^b(s) < \frac{1}{k_s^2}\sum_b e_m^b(s)$;
- $\frac{1}{1+\delta}\frac{1}{k_s^2}\sum_b e_m^b(s) < \sum_i \overline{\mu}^i(s) < \frac{1}{k_s^2}\sum_b e_m^b(s)$.

Let $s \in S^*$. Recall that $\overline{r}_d(s)$ and $\overline{r}(s)$ belong to the real interval $[0,\delta]$. $\sum_i \overline{\delta}^i(s) = \frac{1}{1+\delta}\frac{1}{k_s^1}\sum_b e_m^b(0)$ implies $\overline{r}_d(0) = 0$, thus $\sum_b \overline{d}^b(s) = \frac{1}{k_s^1}\sum_b e_m^b(0)$. But then we have: $\frac{1}{1+\overline{r}_d(s)} = \frac{\sum_i \overline{\delta}^i(s)}{\sum_b \overline{d}^b(s)} = \frac{1}{1+\delta}$, a contradiction.

The proof of the other inequalities is similar, as well for the short term inequalities at $s \in S^*$ as for the long term inequalities.

Claim 3.11. *At the first period, cash-in-advance and end-of-period budget constraints of consumers are saturated.*

Recalling Remark 3.5, this is obvious for the end-of-period constraints of consumers. For banks, we first claim that each $\overline{\pi}^b(0)$ is inferior to its upper bound. Indeed, since each $\overline{\sigma}^i$ and each $\overline{\sigma}^b$ belong respectively to $B^i\big(\overline{\pi}, (\overline{\sigma}^b)_{b\in B}, \overline{\sigma}_{-i}, \overline{\mu}\big)$ and $B^b\big(\overline{\pi}, (\overline{\sigma}^i)_{i\in I}, \overline{\mu}\big)$. Recalling that $\mu_o \geq \frac{1}{1+\delta}$ and adding the end-of-period constraints (13) and (17), we get

$$\sum_i \overline{\pi}^i(0) + \sum_b \pi^b(0) \leq (1+\delta)a + M(0) + M^{CB} < 2[(1+\delta)a + M(0) + M^{CB}].$$

It follows that the end-of-period constraints of banks are saturated. For the cash-in advance constraints, the proof is based on the previous claim.

Claim 3.12. *At the first period, $\sum_{i\in I}\big(\overline{x}^i(0) - e^i(0)\big) = 0$ and $\sum_{i\in I}\overline{z}^i + \sum_{b\in B}\overline{z}^b - z^G = 0$.*

Indeed, since cash-in-advance constraints (12) and (16) are saturated, coming back to the proof of claim 1, we get

$$\overline{p}(0)\cdot\sum_i \big(\overline{x}^i(0) - e^i(0)\big) + \overline{q}\cdot\big(\sum_i \overline{z}^i + \sum_b \overline{z}^b - z^G\big) = 0$$

thus $\overline{p}(0)\cdot\sum_i \big(\overline{x}^i(0) - e^i(0)\big) = 0$ and $\overline{q}\cdot\big(\sum_{i\in I}\overline{z}^i + \sum_{b\in B}\overline{z}^b - z^G\big) = 0$. From $\overline{p}(0) \gg 0$, we deduce $\sum_{i\in I}\big(\overline{x}^i(0) - e^i(0)\big) = 0$. From $\overline{q} \gg 0$, we deduce $\sum_{i\in I}\overline{z}^i + \sum_{b\in B}\overline{z}^b - z^G = 0$.

Remark 3.13. Recalling the behavior of State in the modified truncated economy and Claim 3.9, we get after summing the end-of-period constraints at time $t = 0$ of consumers and banks:

$$\overline{p}(0)\cdot\sum_i\big(\overline{x}^i(0) - e^i(0)\big) + \overline{q}\cdot\big(\sum_i \overline{z}^i + \sum_b \overline{z}^b - z^G\big) = \overline{\mu}_0[M(0) + M^{CB} - \big(\sum_{i\in I}\overline{\pi}^i(0) + \sum_{b\in B}\overline{\pi}^b(0)\big)],$$

which implies: $\sum_{i\in I}\overline{\pi}^i(0) + \sum_{b\in B}\overline{\pi}^b(0) = M(0) + M^{CB}$.

Claim 3.14. *For each $s \in S$, cash-in-advance and end-of-period budget constraints of consumers are saturated.*

For consumers, the proof is as in Claim 3.11. For banks, we first verify that each $\overline{\pi}^b(s)$ is inferior to its upper bound. Indeed, adding the end-of-period constraints (17) and recalling that $\mu_{\geq}\frac{1}{1+d}$ and the previos remark, we get

$$\sum_b \overline{\pi}^b(s) \leq \sum_b e^b_m s + \sum_b \overline{\pi}^b(0) + R(s)\cdot(A+z^G) + \delta(1+\delta)\frac{1}{k_s^2}\sum_b e^b_m(s)$$
$$\leq \sum_b e^b_m s + M(0) + M^{CB} + R(s)\cdot(A+z^G) + \delta(1+\delta)\frac{1}{k_s^2}\sum_b e^b_m(s)$$
$$< M(s) + M(0) + M^{CB} + R(s)\cdot(A+z^G) + \delta(1+\delta)\frac{1}{k_s^2}\sum_b e^b_m(s)$$

the last inequality coming from Assumption **A.4** (2). The rest of the proof is as in Claim 3.11.

Claim 3.15. *For each $s \in S$, $\sum_{i\in I}\big(\overline{x}^i(s) - e^i(s)\big) = 0$.*

Adding the end-of-period 1 constraints at state s of consumers and banks, we get:

$$\overline{p}(s)\sum_{i\in I}\big(\overline{x}^i(s)-e^i(s)\big) = \overline{\mu}_s[M(s)+\sum_{i\in I}\overline{\pi}^i(0)+\sum_{b\in B}\overline{\pi}^b(0)+R(s)\cdot\big(\sum_i \overline{z}^i+\sum_b \overline{z}^b - z^G\big)-(1+\overline{\rho})M^{CB}]$$
$$= \overline{\mu}_s[M(s) + M(0) + M^{CB} - (1+\overline{\rho})M^{CB}].$$

Recalling that $(1+\overline{\rho})M^{CB} \leq M(s) + M(0) + M^{CB}$, we get $\overline{p}(s)\sum_{i\in I}\big(\overline{x}^i(s) - e^i(s)\big) \geq 0$, thus $\overline{p}(s)\sum_{i\in I}\big(\overline{x}^i(s) - e^i(s)\big) = 0$ and, finally $\sum_{i\in I}\big(\overline{x}^i(s) - e^i(s)\big) = 0$.

Claim 3.16. *The two following inequalities hold:*

$$\sum_i\sum_b \overline{v}^i_b < 2[\sum_i e^i_m(0) + (1+\delta)\frac{1}{k_0^2}\sum_b e^b_m(0)],$$
$$\sum_i z^i_- + \sum_b z^b_- < 2A.$$

Adding the cash-in-advance constraints (12) of consumers and recalling that $\overline{\mu}_0 \geq \frac{1}{1+d}$, $\frac{1}{1+\overline{r}(0)} \leq 1$, one gets

$$\sum_i\sum_b \overline{v}^i_b \leq \sum_i e^i_m(0) + (1+\delta)\frac{1}{k_0^2}\sum_b e^b_m(0)$$

which proves the first inequality. Adding the cash in advance constraints (4) and (9) of consumers and banks and taking in account Claim Remark 3.13, one gets

$$R(s)\cdot(\sum_i z^i_- + \sum_b z^b_-) \leq M(0) + M^{CB} + M(s)$$

thus $\sum_i z^i_- + \sum_b z^b_- \leq A < 2A$.

Claim 3.17. *Together with $\frac{1}{1+\overline{r}_d}$, $\frac{1}{1+\overline{r}}$, $\frac{1}{1+\overline{\rho}}$,*

$$\overline{p}'(s) = \frac{\overline{p}(s)}{\overline{\mu}_s}\ \forall s \in S^* \quad \textit{and} \quad \overline{q}' = \frac{\overline{q}}{\overline{\mu}_0}$$

are equilibrium prices for $\mathcal{E}^c$.

We have already seen that $\mu_s \geq \frac{1}{1+\delta}$ for every $s \in S^*$. Dividing by $\overline{\mu}_0$ or $\overline{\mu}_s$ the appropriate budget constraints of the modified budget set restore the initial budget set, which proves the claim. Observe that the equilibrium quantities are $\overline{x}^i$, $\overline{z}^i$, $\overline{z}^b$, $\overline{\delta}^b$, $\overline{\mu}^b$, $\overline{\pi}^i(0)$, $\overline{\pi}^b(0)$, $\overline{\pi}^b(s)$, $\overline{v}^i_b$ are unchanged and that all other equilibrium quantities are divided by the corresponding component of $\overline{\mu}$.

Claim 3.18. *The same prices are equilibrium prices for the original economy.*

As it was proved in different parts of the paper, the bounds defining $\mathcal{E}^c$ are not binding at equilibrium. Since the utility functions of consumers and banks are strictly quasi-concave, it follows that the equilibrium in $\mathcal{E}^c$ is an equilibrium of $\mathcal{E}$. ■

Remark 3.19. It follows from $\mu_s \geq \frac{1}{1+\delta}$ $\forall s \in S^*$ that $\|\overline{p}'(0)\|_1 + \|\overline{q}'\|_1 = \frac{1}{\mu_0} \leq (1+\delta)$ and $\|\overline{p}'(s)\|_1 \leq (1+\delta)$.

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